A New Approach to High-Order Averaging

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Abstract. We present a new approach to perform high-order averaging in oscillatory periodic or quasi-periodic dynamical systems. The averaged system is expressed in terms of (i) scalar coefficients that are universal, i.e. independent of the system under consideration and (ii) basis functions that may be written in an explicit, systematic way in terms of the derivatives of the Fourier coefficients of the vector field being averaged. The coefficients may be recursively computed in a simple fashion. This approach may be used to obtain exponentially small error estimates, as those first derived by Neishtadt for the periodic case and Simó in the quasi-periodic scenario.

Keywords: Averaging, high-order averaging, highly oscillatory problems, Hamiltonian problems, Lie groups, Lie algebras
PACS: 02.30.hq, 42.50pk, 42.20.ij

INTRODUCTION

The papers [1], [2],[3] present a new approach to perform high-order averaging in periodic or quasi-periodic dynamical systems. When this approach is used, the averaged system is expressed in terms of (i) scalar coefficients that are universal, i.e. independent of the system under consideration and (ii) basis functions that may be written in an explicit, systematic way in terms of the derivatives of the Fourier coefficients of the vector field being averaged. The coefficients may be recursively computed in a simple fashion. Thus the averaged system and the associated change of variables are not found here by performing successive changes of variables, as it is the case in conventional approaches.

The new methodology is based on the combinatorial techniques currently used to analyze numerical integrators. It preserves geometric properties and may also be applied to other tasks (e.g. to find formal conserved quantities in Hamiltonian problems, see [2]).

FORMAL RESULTS

Assume that the problem to be averaged has been rewritten [4], [5] to take the familiar format:

$$\frac{d}{dt}y = \varepsilon f(y, t\omega), \tag{1}$$

$$y(0) = y_0 \in \mathbb{R}^D, \tag{2}$$

where ε is a small parameter, $f = f(y,\theta)$ is smooth and 2π-periodic in each of the components θ^j, $j = 1,\ldots,d$, of θ, i.e. $\theta \in \mathbb{T}^d$, and $\omega \in \mathbb{R}^d$ is a constant vector of angular frequencies assumed to be *non-resonant*, i.e. $\mathbf{k}\cdot\omega \neq 0$, for each $\mathbf{k} \in \mathbb{Z}^d$. When $d = 1$ the right-hand side of (1) is *periodic* in the variable t; for $d > 1$ the time-dependence is *quasi-periodic*. We denote by $f_{\mathbf{k}}$ the Fourier coefficients of $f(y,\theta)$ so that

$$f(y,\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{i\mathbf{k}\cdot\theta} f_{\mathbf{k}}(y). \tag{3}$$

As shown in [2], (1) may be quasi-stroboscopically averaged to get a autonomous system

$$\frac{d}{dt}Y = \varepsilon F(Y,\varepsilon), \quad F(Y) = F_1(Y) + \varepsilon F_2(Y) + \cdots + \varepsilon^{n-1} F_n(Y) + \cdots \tag{4}$$

The functions F_n are explicitly given in terms of the commutators (Lie brackets) of the functions $f_{\mathbf{k}}$ in (3) as

$$F_n(y) = \sum_{\mathbf{k}_1,\dots,\mathbf{k}_n\in\mathbb{Z}^d} \frac{1}{j}\,\bar{\beta}_{\mathbf{k}_1\cdots\mathbf{k}_n}\,[[\cdots[[f_{\mathbf{k}_1}, f_{\mathbf{k}_2}], f_{\mathbf{k}_3}]\cdots], f_{\mathbf{k}_n}](y), \tag{5}$$

where the scalar *coefficients* $\bar{\beta}_{\mathbf{k}_1\cdots\mathbf{k}_n}$ are *universal*, i.e. while they depend on ω, they are independent of $f(y,\theta)$ and may be computed by means of simple recursions. The solution of (1)–(2) as a formal series may be written as

$$y(t) = U(Y(t), t\omega, \varepsilon),$$

where Y is the solution of (4) with initial condition $Y(0) = y_0$ and U is a change of variables parameterized by $\theta \in \mathbb{T}^d$

$$y = Y + \varepsilon\check{U}(Y,\theta,\varepsilon); \quad \check{U}(Y,\theta,\varepsilon) = u_1(Y,\theta) + \cdots + \varepsilon^{n-1}u_n(Y,\theta) + \cdots \tag{6}$$

The functions u_n (as the functions F_n in (5)) may be computed explicitly in terms of the $f_{\mathbf{k}}$ and of a family of scalar universal coefficients κ that (as the coefficients $\bar{\beta}$ for F_n) may be computed recursively.

The fact that (5) is constructed in terms of commutators implies that if the Fourier coefficients $f_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d$, belong to a specific Lie subalgebra of the Lie algebra of vector fields (e.g. each $f_{\mathbf{k}}$ is Hamiltonian) then the quasi-stroboscopic averaged system (4) will also belong to the same Lie subalgebra (e.g. averaging a Hamiltonian system will lead to a Hamiltonian system). Similarly the change of variables (6) will belong to the corresponding Lie group (in the example, the change of variables will be canonical).

ERROR ESTIMATES

It is well known that the series in (4) and (6) are in general divergent and have to be truncated in order to approximate the solution y of (1)–(2). Neishtadt [6] and Simó [7] have proved that those truncations may be performed so as to yield errors that are exponentially small in ε. The present technique is very well suited to derive such estimates as shown in [3] for the periodic ($d = 1$) case. Here we deal with the quasi-periodic ($d > 1$) case. We assume that f satisfies:

Assumption A. There exist $R > 0$, $\mu > 0$ and an open set $\mathscr{U} \supset \mathscr{K}_R$, such that, for each $\theta \in \mathbb{T}^d$, $f(\cdot,\theta)$ may be extended to a map $\mathscr{U} \to \mathbb{C}^D$ that is analytic at each point $y \in \mathscr{K}_R$. Furthermore the Fourier coefficients $f_{\mathbf{k}}$ of f have bounds

$$\forall \mathbf{k} \in \mathbb{Z}^d, \qquad \|f_{\mathbf{k}}\|_R \le a_{\mathbf{k}} e^{-\mu|\mathbf{k}|}, \qquad a_{\mathbf{k}} \ge 0,$$

where the $a_{\mathbf{k}}$ are such that

$$M = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} < \infty.$$

An additional hypothesis required to deal with small denominators that appear in the recursions for the coefficients $\bar{\beta}$ and κ is the assumption that the vector $\omega \in \mathbb{R}^d$ satisfies a *strong non-resonance* condition

$$\forall \mathbf{k} \in \mathbb{Z}^d\backslash\{\mathbf{0}\}, \qquad |\mathbf{k}\cdot\omega| \ge c|\mathbf{k}|^{-\nu} \tag{7}$$

for some constants $c > 0$ and $\nu > 0$.

We can then prove:

Theorem 1 *Suppose that f satisfies the requirements in Assumption A and ω satisfies the condition (7). The application of the truncated change of variables*

$$y = Y + \varepsilon\check{U}^{(N)}(Y,t,\varepsilon)$$

with

$$\check{U}^{(N)}(y,\theta,\varepsilon) = u_1(y,\theta) + \varepsilon u_2(y,\theta) + \cdots + \varepsilon^{N-1}u_N(y,\theta)$$

and

$$|\varepsilon| \le \varepsilon_0, \qquad \varepsilon_0 = \varepsilon_0(N) = \frac{R}{4LN^{\nu+1}}, \qquad L = \frac{2M\nu^\nu}{c\,\mu^\nu e^\nu},$$

to the initial value problem (1)–(2) results in a problem

$$\frac{d}{dt}Y = \varepsilon\left(F^{(N)}(Y,\varepsilon)+R^{(N)}(Y,t,\varepsilon)\right), \qquad Y(0)=y_0,$$

where

$$F^{(N)}(y,\varepsilon)=F_1(y)+\varepsilon F_2(y)+\cdots+\varepsilon^{N-1}F_N(y)$$

(the functions F_j are as defined in (5)). The remainder $R^{(N)}$ possesses the bound

$$\|R^{(N)}(\cdot,t,\varepsilon)\|_{R/2} \leq \frac{5|\varepsilon/\varepsilon_0|^N}{1-|\varepsilon/\varepsilon_0|}M.$$

In particular, assume that for given ε, with $|\varepsilon| \leq R/(4eL)$, N is chosen as the integer part of the real number $\left(R/(4eL|\varepsilon|)\right)^{1/(\nu+1)} \geq 1$. Then the following exponentially small estimate holds true:

$$\|R^{(N)}(\cdot,\theta,\varepsilon)\|_{R/2} \leq \frac{5e^2}{e-1}M\exp\left(-\frac{K}{|\varepsilon|^{1/(\nu+1)}}\right), \qquad K=\left(\frac{R}{4eL}\right)^{1/(\nu+1)}.$$

ACKNOWLEDGMENTS

P. Chartier has received financial support from INRIA through the associated team MIMOL. A. Murua and J.M. Sanz-Serna have been supported by projects MTM2010-18246-C03-03 and MTM2010-18246-C03-01 respectively from Ministerio de Ciencia e Innovación.

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