h-dependent stability thresholds avoid the need for a priori bounds in nonlinear convergence proofs.

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Abstract. We show that the use of the concept of stability with h-dependent thresholds introduced by López-Marcos and Sanz-Serna avoids the need for a priori bounds in convergence proofs of discretizations of nonlinear problems in partial differential equations. It is also shown that h-dependent thresholds are indeed necessary in the investigation of the stability of such discretizations.

1. Introduction.

In [5] and [6] López-Marcos and Sanz-Serna have suggested a definition of stability for numerical methods for nonlinear problems. Their definition is based on so called h-dependent stability thresholds. In the present paper we show that the use of the formalism developed in [7], [5], [6], [4] may avoid the need for a priori estimates in nonlinear convergence proofs. Although the ideas in this paper are very general, we present them as applied to a particular case. Namely our treatment follows the thesis [1] which analyzes several finite-difference and spectral schemes for the nonlinear Dirac equation

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + i\, f(|u_1|^2 - |u_2|^2)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.1}$$

where u_1, u_2 are complex-valued functions of x and t and f is a smooth real-valued function of a real variable. Systems of the form (1.1) have been used to model several physical situations (see references in [1]); in such applications f is typically a polynomial and therefore is not globally Lipschitz continuous.

To simplify the exposition we consider, instead of (1.1), the single model equation

$$u_t = u_x + i\, f(|u|^2)\, u, \tag{1.2}$$

with u complex valued, f C^1-continuous. While it is straightforward to extend the results presented here for (1.2) to the physically relevant system (1.1), working with (1.2) has the advantage of allowing a notation more compact than that required for (1.1).

For simplicity, we only consider the 1-periodic initial value problem for (1.2), specified by the conditions

$$u(x, t) = u(x+1, t), \quad -\infty < x < \infty, \quad 0 \le t \le T < \infty, \tag{1.3}$$

$$u(x, 0) = q(x), \quad -\infty < x < \infty, \tag{1.4}$$

with q a given 1-periodic function. (For nonperiodic boundary condition see [1].) We assume throughout that (1.2)-(1.4) has a unique smooth solution u.

In section 2 we present a numerical method for (1.2)-(1.4) and recall the standard technique for proving its convergence, underlining the need for a priori estimates. In section 3, we briefly outline the general formalism of López-Marcos and Sanz-Serna and show how the main result in López-Marcos [4] (see also [5], [6]) can be applied to eliminate several steps from standard convergence proofs.

The formalism in [5], [6], [4] is built around a notion of stabilty with h-dependent thresholds. The idea of stabilty thresholds, useful as it is, does not appear to be well known. The final section 4 takes up this issue and proves that h-dependent stability thresholds are indeed necessary when dealing with discretizations of nonlinear partial differential equations.

2. Numerical method and standard analysis.

We introduce a uniform grid $x_j = jh$, $j = \dots, -2, -1, 0, 1, 2, \dots$; $h = 1/J$, J an integer, and the time-levels $t_n = nk$, $n = 0, 1, \dots, N$; $N = [T/k]$; $k = ch$, c a fixed positive number. The equation (1.2) is discretized as follows

$$(2.1)\qquad (1/k)(U_j^{n+1} - U_j^n) = [1/(4h)](U_{j+1}^{n+1} - U_{j-1}^{n+1}) + [1/(4h)](U_{j+1}^n - U_{j-1}^n) + g((1/2)(U_j^{n+1} + U_j^n));\quad j = 1, 2, \dots, J;\quad n = 0, 1, \dots, N-1;$$

where, by periodicity, $U_0^n = U_J^n$, $U_{J+1}^n = U_1^n$, $n = 0, 1, \dots, N$, and g is the mapping defined by $g(v) = i\,f(|v|^2)v$, $v \in \mathbb{C}$. Other finite-difference and spectral methods for (1.2) have been studied in [1], but (2.1) suffices to illustrate the points we want to make.

If we consider the vectors $\mathbf{U}^n = [U_1^n, U_2^n, \dots, U_J^n]^T \in \mathbb{C}^J$, $n = 0, 1, \dots, N$, then the equations (2.1) take the form

$$(2.2a)\qquad (1/k)(\mathbf{U}^{n+1} - \mathbf{U}^n) = L_h[(1/2)(\mathbf{U}^{n+1} + \mathbf{U}^n)] + \mathbf{G}((1/2)(\mathbf{U}^{n+1} + \mathbf{U}^n));\quad n = 1, 2, \dots, N-1;$$

where L_h is a JxJ, skew-symmetric matrix and $\mathbf{G}$ is the $\mathbb{C}^J$-valued mapping defined by $\mathbf{G}(\mathbf{V}) = [g(V_1), g(V_2), \dots g(V_J)]^T$, if $\mathbf{V} = [V_1, V_2, \dots, V_J]^T \in \mathbb{C}^J$. The recursion (2.2a) is supplemented by an initial condition

$$(2.2b)\qquad \mathbf{U}^0 = \mathbf{q}_h,$$

with $\mathbf{q}_h$ a given approximation to $[\,q(x_1), q(x_2), \dots, q(x_J)\,]^T$.

Now let $\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N$ and $\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N$ be two sequences of J-dimensional complex vectors. When these vectors are substituted in the relations (2.2) that define the numerical method, they originate residuals $\mathbf{F}^n$, $\mathbf{G}^n$ as follows

$$(2.3)\qquad \mathbf{F}^n = (1/k)(\mathbf{V}^{n+1} - \mathbf{V}^n) - L_h[(1/2)(\mathbf{V}^{n+1} + \mathbf{V}^n)] - \mathbf{G}((1/2)(\mathbf{V}^{n+1} + \mathbf{V}^n));\quad n = 0, 1, \dots, N-1,$$

$F^0 = \mathbf{V}^0 - \mathbf{q}_h$;

(2.4) $\mathbf{G}^n = (1/k)(\mathbf{W}^{n+1} - \mathbf{W}^n) = L_h[(1/2)(\mathbf{W}^{n+1} + \mathbf{W}^n)] + \mathbf{G}((1/2)(\mathbf{W}^{n+1} + \mathbf{W}^n))$;

$n = 0, 1, \ldots, N-1$,

$\mathbf{G}^0 = \mathbf{W}^0 - \mathbf{q}_h$.

The stability question [7] is that of bounding the norms of the vectors $\mathbf{V}^n - \mathbf{W}^n$ in terms of the norms of the vectors $\mathbf{F}^n - \mathbf{G}^n$, <u>uniformly</u> in h, k. Here we employ the standard L^2-norm

$$\| \mathbf{V} \|^2 = \sum_{j=1}^{J} h | V_j |^2, \qquad \mathbf{V} = [V_1, V_2, \ldots, V_J]^T.$$

Of course, the main application of the idea of stability lies in the proof of convergence. There the vectors $\mathbf{V}^n$ are chosen to be the numerical solutions $\mathbf{U}^n$, n =0, 1, ..., N, defined in (2.2), so that $\mathbf{F}^n = \mathbf{0}$, and the vectors $\mathbf{W}^n$ are taken to be the vectors $\mathbf{u}^n$ of grid-restriction of the theoretical solution, $\mathbf{u}^n = [u(x_1, t_n), u(x_2, t_n), \ldots, u(x_J, t_n)]^T$, n = 0, 1, ..., N, a choice which entails that the corresponding residuals $\mathbf{G}^n$ are vectors containing the truncation errors. Thus the notion of stability makes it possible to bound the errors $\| \mathbf{U}^n - \mathbf{u}^n \|$ in terms of the truncation errors. For simplicity, round-off errors are not taken into account in this paper. (In the presence of round-off [7] the computed vectors $\mathbf{U}^n$ do not satisfy (2.2) exactly and accordingly they originate nonzero residuals $\mathbf{F}^n$ which would feature in the bound for $\| \mathbf{U}^n - \mathbf{u}^n \|$.)

Let us return to the stabilty question, where $\mathbf{V}^n$ and $\mathbf{W}^n$ are not necessarily equal to $\mathbf{U}^n$ and $\mathbf{u}^n$. To prove stability, the standard approach in evolutionary problems starts by subtracting (2.4) from (2.3) to arrive at

(2.5) $[I - (k/2)L_h]\mathbf{E}^{n+1} = [I + (k/2)L_h]\mathbf{E}^n + k[\mathbf{F}^{n+1} - \mathbf{G}^{n+1}]$

$+ k[\mathbf{G}((1/2)(\mathbf{V}^{n+1} + \mathbf{V}^n)) - \mathbf{G}((1/2)(\mathbf{W}^{n+1} + \mathbf{W}^n))]$;

$n = 0, 1, \ldots, N-1$;

(2.6) $\mathbf{E}^0 = \mathbf{F}^0 - \mathbf{G}^0$,

where I denotes the JxJ identity matrix and we have used the abbreviation $\mathbf{E}^n := \mathbf{V}^n - \mathbf{W}^n$, n = 0, 1, ..., N. Since L_h is skew-symmetric

$$\| (I - (k/2)L_h)^{-1} \| \le 1, \quad \| (I - (k/2)L_h)^{-1} (I + (k/2)L_h) \| = 1,$$

and, on taking norms in (2.5), we conclude

(2.7) $$\| E^{n+1} \| \leq \| E^n \| + k\| F^{n+1} - G^{n+1} \| + \| \mathbf{G}((1/2)(\mathbf{V}^{n+1} + \mathbf{V}^n)) - \mathbf{G}((1/2)(\mathbf{W}^{n+1} + \mathbf{W}^n)) \|;$$

$$n = 0, 1, \ldots, N-1.$$

To deal with the nonlinear term in (2.7) we would like to employ a Lipschitz bound. However, g is not globally Lipschitz in $\mathbb{C}$ and therfore **G** is not globally Lipschitz in $\mathbb{C}^J$. On the other hand, g has been assumed to be smooth and therefore for each positive K, there exists a constant L = L (K) such that

$$|g(v) - g(w)| \leq L\,|v - w|, \quad \text{if } |v|, |w| < K.$$

This implies

(2.8) $$\| \mathbf{G}(\mathbf{V}) - \mathbf{G}(\mathbf{W}) \| = \Big(\sum_{i=1}^{J} h |g(V_i) - g(W_i)|^2 \Big)^{1/2} \leq \Big(\sum_{i=1}^{J} hL^2 |V_i - W_i|^2 \Big)^{1/2} = L\| \mathbf{V} - \mathbf{W} \|$$

<u>if</u>

(2.9) $$|V_i|, |W_i| < K, \quad i = 1, 2, \ldots, J.$$

Note that the last condition holds if

$$\| \mathbf{V} \|, \| \mathbf{W} \| < Kh^{1/2},$$

but that the requirement

$$\| \mathbf{V} \|, \| \mathbf{W} \| < K$$

is not enough to guarantee (2.9).

Thus, to make some progress in (2.7), it is not possible to let $\mathbf{V}^n$, $\mathbf{W}^n$ be arbitrariry in $\mathbb{C}^J$ and the attention must be restricted to values of $\mathbf{V}^n$, $\mathbf{W}^n$ near the theoretical vector $\mathbf{u}^n$. Namely we <u>assume</u> that $\mathbf{V}^n$, $\mathbf{W}^n$ satisfy

(2.10) $$\| \mathbf{V}^n - \mathbf{u}^n \| < Rh^{1/2}, \quad \| \mathbf{W}^n - \mathbf{u}^n \| < Rh^{1/2}, \quad n = 0, 1, \ldots, N,$$

where R is an arbitrarily chosen positive constant. The hypothesis (2.10) guarantees that (2.9) holds for $\mathbf{V}^n$, $\mathbf{W}^n$, $n = 0, 1, \ldots, N$, with $K = R + \max\{|u(x, t)| : 0 \leq x \leq 1, 0 \leq t \leq T\}$. Then, the use of (2.8) in (2.7), yields

$$\| E^{n+1} \| \leq \| E^n \| + k\| F^{n+1} - G^{n+1} \| + k(L/2)(\| E^{n+1} \| + \| E \|),$$

$$n = 0, 1, \ldots, N-1,$$

or

$$(1 - (kL/2)) \| \mathbf{E}^{n+1} \| \leq (1 + (kL/2)) \| \mathbf{E}^n \| + k \| \mathbf{F}^{n+1} - \mathbf{G}^{n+1} \|,$$

$$n = 0, 1, \ldots, N-1.$$

If we now restrict the attention to values of k such that $kL < 1$, so that

$$(1 - (kL/2))^{-1}(1 + (kL/2)) \leq \exp(3kL/2),$$

then a simple induction argument shows that

$$(2.11) \qquad \max_{0 \leq n \leq N} \| \mathbf{V}^n - \mathbf{W}^n \| \leq S \{ \| \mathbf{F}^0 - \mathbf{G}^0 \| + \sum_{n=1}^{N} k \| \mathbf{F}^n - \mathbf{G}^n \| \}$$

with

$$(2.12) \qquad S = \exp(3LT/2).$$

To sum up we have proved

Proposition (Stabilty). Given $R > 0$ and vectors $\mathbf{V}^n$, $\mathbf{W}^n$, $n = 0, 1, \ldots, N$, satisfying (2.10), suppose that $kL < 1$, where L is the Lipschitz constant of g in the disk with centre at 0 and radius $R + \| u \|_\infty$. Then (2.11) holds with S given by (2.12).

Taking up the issue of convergence, it is clear that the proposition together with the consistence of the scheme do not directly imply convergence [6]. In fact, before we can use the bound (2.11) with the numerical solution playing the role of $\mathbf{V}$ and the thoretical solution playing the role of $\mathbf{W}$, we must check:

(i) That the numerical solutions exist at all, i.e. that, at least for k small, the recursion (2.2) defines a sequence of vectors $\mathbf{U}^n$, $n = 0, 1, \ldots, N$.

(ii) That the vectors $\mathbf{U}^n$ satisfy the hypothesis of the proposition, namely that the following a priori bound holds

$$(2.13) \qquad \| \mathbf{U}^n - \mathbf{u}^n \| < Rh^{1/2}, \quad n = 0, 1, \ldots, N,$$

for a suitable constant $R > 0$.

Thus, generally speaking, standard convergence proofs of nonlinear algorithms for steady or evolutionary problems consist of the following steps:

1) Proof of consistency. This bounds the residual originated in the discrete equations by the theoretical solution.

2) Proof of a stability proposition like the one above. This bound the difference between two elements $\mathbf{V}$ and $\mathbf{W}$ in the neighbourhood of the theoretical solution in terms of the difference between the residuals they originate in the discrete equations. (Often this step is only carried out for the particular choice where $\mathbf{V}$ and $\mathbf{W}$ are

respectively the numerical and theoretical solutions.)

3) Proof of the existence of the discrete solution.

4) Proof of an a priori bound which guarantees that the numerical solution is not far from the theoretical solution and makes it possible to take advantage of the local properties of the nonlinear terms in the neighbourhood of the theoretical solution.

In many cases the proofs of the steps 3) and 4) turn out to be more involved than the proof of the stability bound itself. We are going to show next that often steps 3) and 4) above are not really necessary.

In fact we are going to describe a formalism developed by López-Marcos and Sanz-Serna [7], [5], [6], [4] where the checking of 1) and 2) often implies the convergence. The formalism will be presented in an abstract, general way, but we shall simultaneously outline its application to the scheme (2.2).

3. A general formalism.

In the formalism all the relations defining a numerical scheme are rewriten in the abstrac form

(3.1) $$\Phi_h(\mathcal{U}_h) = \mathcal{O},$$

where $\mathcal{U}_h$ collects all numerical results and Φ_h is a mapping with domain $D_h \subset X_h$ and values in Y_h, with X_h and Y_h normed spaces such that

(3.2) $$\dim X_h = \dim Y_h < \infty.$$

The parameter h takes values in a set H with inf H = 0.

Example. For the scheme (2.2)

$$\mathcal{U}_h = [\,\mathbf{U}^{0T}, \mathbf{U}^{1T}, \ldots, \mathbf{U}^{NT}\,]^T \in (\mathbb{C}^J)^{N+1}$$

and Φ_h is given by

$$\Phi_h(\mathcal{V}_h) = \begin{bmatrix} \mathbf{V}^0 - \mathbf{q}_h \\ k^{-1}(\mathbf{V}^1 - \mathbf{V}^0) - L_h((1/2)(\mathbf{V}^1 + \mathbf{V}^0)) - \mathbf{G}((1/2)(\mathbf{V}^1 + \mathbf{V}^0)) \\ \cdots \\ k^{-1}(\mathbf{V}^N - \mathbf{V}^{N-1}) - L_h((1/2)(\mathbf{V}^N + \mathbf{V}^{N-1})) - \mathbf{G}((1/2)(\mathbf{V}^N + \mathbf{V}^{N-1})) \end{bmatrix}$$

if $\mathcal{V}_h = [\,\mathbf{V}^{0T}, \mathbf{V}^{1T}, \ldots, \mathbf{V}^{NT}\,]^T \in (\mathbb{C}^J)^{N+1}$.

Thus (3.1) represents the set of relations (2.2). The normed space X_h is $(\mathbb{C}^J)^{N+1}$ with the norm

(3.3) $$\| \mathcal{V}_h \|_{X_h} = \max_{0 \le n \le N} \| \mathbf{V}^n \|, \quad \mathcal{V}_h = [\, \mathbf{V}^{0T}, \mathbf{V}^{1T}, \dots, \mathbf{V}^{NT} \,]^T$$

and Y_h is the space $(\mathbb{C}^J)^{N+1}$ with the norm

(3.4) $$\| \mathcal{F}_h \|_{Y_h} = \| \mathbf{F}^0 \| + \sum_{1 \le n \le N} k \| \mathbf{F}^n \|.$$

In (3.3), (3.4) $\| \cdot \|$ denote the L^2-norm in $\mathbb{C}^J$. The condition (3.2) is trivially satisfied. Here the domain D_h is the whole of X_h, but situations where D_h is smaller than X_h would arise if the nonlinear function g in (2.1) were not defined in the whole of $\mathbb{C}$. The parameter h ranges the set H of numbers of the form $1/J$, J a positive integer and k is not viewed as an indepedent parameter but as the function of h given by $k = ch$.□

The abstract discretization (3.1) is said to be consistent if $\|\Phi_h(\mathcal{U}_h)\|_{Y_h} \to 0$ as $h \to 0$, where $\mathcal{U}_h \in D_h$ is a suitable representation of the theoretical solution (say a grid restriction or a projection).

Example. For the scheme (2.2), the standard Taylor expansions show that $\|\Phi_h(\mathcal{U}_h)\|_{Y_h} = O(h^2)$, as $h \to 0$, provided that the initial approximations $\mathbf{q}_h$ satisfy

(3.5) $$\| \mathbf{q}_h - \mathbf{u}^0 \| = O(h^2), \quad h \to 0. \square$$

We say that the discretization (3.1) is stable if there exist $h_0 > 0$ and $S > 0$, and values R_h, $0 < R_h \le \infty$, such that $h \le h_0$, $\mathcal{V}, \mathcal{W} \in X_h$, $\| \mathcal{V} - \mathcal{U}_h \|_{X_h} \le R_h$, $\| \mathcal{W} - \mathcal{U}_h \|_{X_h} \le R_h$, imply that

(3.6) $$\| \mathcal{V} - \mathcal{W} \|_{X_h} \le S \| \Phi_h(\mathcal{V}) - \Phi_h(\mathcal{W}) \|_{Y_h}.$$

The constant S is the stability constant and the values R_h are called the stability thresholds.

Example. In the case of the scheme (2.2), the choices of norms in (3.3)-(3.4) and the definition of $\mathbf{F}^n$ and $\mathbf{G}^n$ in (2.3)-(2.4) imply that the stability bound (3.6) is identical with (2.11). Therefore the proposition in the previous section shows the stability of (2.2) with

$h_0 = (cL)^{-1}$, $R_h = Rh^{1/2}$ and S given in (2.12).□

The key result of the formalism is the following theorem [4], [5], which is based on a deep result due to Stetter ([9], Lemma 1.2.1):

Theorem. Assume that : (i) (3.1) is stable with thresholds R_h. (ii) The mapping Φ_h is defined and continuos in the ball $B(u_h, R_h)$ with centre at u_h and radius R_h. (iii) (3.1) is consistent and

(3.7) $$\|\Phi_h(u_h)\|_{Y_h} = o(R_h), \quad h \to 0.$$

Then, for h sufficiently small, the equation (3.1) has a unique solution U_h in the ball $B(u_h, R_h)$ and

$$\| U_h - u_h \|_{X_h} = O(\| \Phi_h(u_h) \|_{Y_h}), \quad h \to 0.$$

Example. For (2.2), under the condition (3.5), the hypotheses of the theorem cleary hold, since in (3.7), $R_h = Rh^{1/2}$ and $\|\Phi_h(u_h)\|_{Y_h} = O(h^2)$. Therefore we conclude that, for h small, the equations (2.2a) posses solutions U^n, $n = 1, 2, \ldots, N$, and that those solutions are locally unique. Furthermore

$$\max_{0 \le n \le N} \| U^n - u^n \| = O(h^2).$$

Thus the application of the abstract theorem above has made it possible to prove the convergence of (2.2) without a separate investigation of the existence of discrete solutions and without any need for a priori bounds. □

Remarks. 1) The formalism above is quite general and has been applied to finite differences, finite elements and spectral methods; boundary-value, initial-value, and initial-boundary-value problems [5], [6], [4], [1].

2) An appealing feature of the definition of stability employed here is that, with respect to this definition, a smooth, nonlinear discretization is stable if and only if its linearization around the theoretical solution is a stable (linear) discretization [6]. This result makes it possible to investigate the stability of nonlinear algorithms by using linear techniques.

3) Consider the case where the discrete equations take the form of a linear recurrence

$\mathbf{U}^0$ given

$\mathbf{U}^{n+1} = C(h)\mathbf{U}^n$, $n = 0, 1, \ldots, [T/h] = N$,

where the elements $\mathbf{U}^n$ belong to a linear space B_h and $C(h)$ is a linear mapping in B_h. With the choice of norms in (3.3)-(3.4), where now $\| \, . \, \|$ represents the norm in B_h, a necessary and sufficient condition for stability as defined here [7], [5], [8] is given by the familiar requirement

$$\sup \{ \|C(h)^n\| : h \in H, nh \leq T \} < \infty.$$

Thus, the notion of stability employed here contains as a particular case the celebrated definition introduced by Lax.

Note that this remark also shows the relevance of the norms (3.3)-(3.4) used in our analysis of (2.2).

4. h-depent stability thresholds.

The main feature of the definition of stability used in this paper is the presence of h-dependent stability thresholds. Many authors define stability without resorting to thresholds, i.e. they demand that (3.6) be verified for arbitrary $\mathcal{V}$, $\mathcal{W}$ in X_h (h small). Such definitions without thresholds, are utterly unsuitable for the investigation of most nonlinear situations [6], [4].

Stetter [9] and Keller [3] have given definitions of stability with thresholds, but in their theories the thresholds are not allowed to depend on h. Guo Ben-Yu [2] uses a definition with h-dependent thresholds which is different from the concept employed in this paper in that in his theory the threshold condition is imposed on the residuals $\mathcal{F}$ and $\mathcal{G}$ rather that on the elements $\mathcal{V}$ and $\mathcal{W}$. A comparison between these definition of stability has been presented in [5], [4].

Keller's definition is identical to that introduced by López-Marcos and Sanz-Serna [5], [6], [4] and given in section 3, except for the fact that Keller demands that the thresholds R_h be independent of h. In this section we are going to prove that the scheme (2.2), which we have shown to be convergent, is not stable in the sense of Keller, thus implying that the introduction of h-dependent thresholds is indeed necessary.

It is enough to consider the particular case where in (1.2), $f(s) = s^2$. The method of characteristics reveals that to each 1-periodic initial datum q there corresponds a unique solution of (1.2)-(1.4) which we denote by Eq. Explicity

$$Eq(x, t) = \exp(i\,|q(x+t)|^4 t)q(x+t).$$

Note that the evolution in time consists of a displacement along the characteristics along with a complex rotation with frecuency $|q(x+t)|^4$. Cleary Eq is a C^3 function of x and t if $q \in C^3$ and then the results of the previous section imply that the scheme (2.2) converges at a rate $O(h^2)$, provided that $q \in C^3$. We argue <u>ad absurdum</u> and suppose that (2.2) is Keller stable in the particular case $q \equiv 0$ ($u \equiv 0$). Then, by definition , there exists constants R and S such that, for h small, $\|\mathbf{V}^n\| < R$, $\|\mathbf{W}^n\| < R$, $n = 0, 1, \ldots, N$, imply (2.11). By convergence, it is easy to see, that if ρ, σ are C^3-continous, 1-periodic initial data, then

(4.1) $\quad \|\rho\| < R,\ \|\sigma\| < R \quad$ imply $\quad \max_{0\le t\le T} \|E\rho(\cdot, t) - E\sigma(\cdot, t)\| \le S\|\rho - \sigma\|$,

where the norm is the standard norm in $L^2(0, 1)$. In fact (4.1) follows from (2.1) by setting $\mathbf{V}^n$ and $\mathbf{W}^n$ equal to the numerical solution given by the scheme for the initial data ρ and σ, and then letting $h \to 0$.

Now choose, for each m = 1, 2, ..., initial conditions ρ_m and σ_m as follows

$|\rho_m(x)| \le m,\ |\sigma_m(x)| \le m - 1,\ 0 \le x \le 1,$

$\rho_m(x) = m,\ \sigma_m(x) = m - 1,\ |x - 1/2| \le R^2/(4m^2),$

$\rho_m(x) = \sigma_m(x) = 0,\ 0 \le x \le 1,\ |x - 1/2| > R^2/(4m^2) + R^3/(4m^3),$

ρ_m and σ_m indefinitly differentiable.

It is trivial to check that $\|\rho_m\| < R$, $\|\sigma_m\| < R$. Furthermore $\|\rho_m - \sigma_m\| \to 0$ as $m \to 0$, because the support of $\rho_m - \sigma_m$ shrinks with increasing m and, in that support $\rho_m - \sigma_m$ is, essentially, 1. Therfore, by (4.1)

(4.2) $\quad \max_{0\le t\le T} \|E\rho_m(\cdot, t) - E\sigma_m(\cdot, t)\| \to 0, \qquad m \to 0.$

On the other hand the complex rotation experienced by ρ_m in the time-evolution is faster than that experienced by σ_m and, accordingly, the phases of $E\rho_m$ and $E\sigma_m$, which coincide at time t = 0, later become opposite to each other. This happens at time $t_m = \pi / (m^4 - (m-1)^4)$ which is clearly $\leq T$ for m large.

Then

$$\| E\rho_m(\cdot, t_m) - E\sigma_m(\cdot, t_m) \|$$

$$\geq \int_{|1/2 - t| \leq R^2/4m^2} | \exp(im^4 t_m)m - \exp(i(m-1)^4 t_m)(m-1) |^2 dx$$

$$= (R^2/2)\,((4m^2 - 4m + 1)/m^2) > R^2.$$

On comparing the last relation with (4.2) we obtain the sought contradition.

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