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A note on uniform in time error estimates for approximations to reaction-diffusion equations

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The approximation of solutions of reaction-diffusion equations that approach asymptotically stable, hyperbolic equilibria is considered. Near such equilibria trajectories of the equation contract and hence it is possible to seek error estimates that are uniformly valid in time. A technique for the derivation of such estimates is illustrated in the context of an explicit Euler finite-difference scheme.

1. Introduction

Classical error estimates for time-dependent problems on an interval $0<t<\tau$ involve an error constant that grows like e^{τ}. This reflects the fact that, in general, trajectories of well-posed initial-value problems may diverge at an exponential rate in time and the numerical solution can be regarded as a trajectory starting close to the true trajectory in phase space. When the differential equation has some structure which forces trajectories to contract, it is sometimes possible to derive error estimates which are uniformly valid in time. See Stetter (1973), Chapters 3.5 and 4.6, for applications of this idea to the numerical solution of ordinary differential equations. See also Dekker and Verwer (1984), Chapters 1 and 2.

In this note we are interested in the derivation of uniform in time error estimates for approximations to reaction-diffusion problems such as

$$u_t = u_{xx} + f(u), \qquad 0<x<1, \qquad t>0, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t>0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad 0<x<1, \tag{1.3}$$

a question first considered by Hoff (1978). In sections 5 and 6 of Hoff (1978) conditions are enforced which ensure that (1.1–1.3) has a *unique* equilibrium within some invariant region S and that trajectories contract within S. Under these conditions Hoff proves uniform in time error estimates, assuming that the initial data $u_0(x)$ is in S. Hoff's ideas were used by Galeone (1983) to study the decay of numerical solutions to spatially homogeneous solutions of (1.1), subject to homogeneous Neumann boundary conditions.

It is possible to improve considerably upon the results in Hoff (1978): it is necessary only to assume the existence of an asymptotically stable, hyperbolic equilibrium (which need not necessarily be unique). Trajectories then contract only in some neighbourhood of the equilibrium. However, standard finite time error estimates can be used to ensure that the numerical solution enters this neighbourhood, provided that the true solution does; contractivity then takes over. This idea was used by Larsson (1989) in his study of finite element approximations of scalar reaction-diffusion equations. See also Heywood and Rannacher (1986) for an application of this approach to the Navier–Stokes equations. Khalsa (1977) considers a particular reaction-diffusion equation and analyses stability properties of a discrete problem by use of the Conley index. It is an interesting and open question as to whether the important work of Beyn (1987) on the uniform in time approximation of trajectories near *unstable* equilibria (which hinges on the approximation of stable and unstable manifolds) can be extended from ordinary to partial differential equations.

Our purpose here is to present a simple argument whereby uniform in time error estimates can be obtained for trajectories approaching a stable equilibrium. The argument applies to a wide variety of implicit and explicit time-stepping schemes. In contrast, Khalsa (1987) considers only time-continuous discretisations and Larsson (1989) analyses only the dissipative backward Euler scheme.

For simplicity, we shall limit our analysis to the forward Euler method with central differences in space:

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}=\frac{\delta^2 U_j^n}{\Delta x^2}+f(U_j^n), \quad j=1,\ldots,J-1, \quad n=0,1,2,\ldots, \tag{1.4}$$

$$U_0^n=U_J^n=0, \quad 1,2,\ldots, \tag{1.5}$$

$$U_j^0=u_0(j\,\Delta x), \quad j=0,\ldots,J, \tag{1.6}$$

where U_j^n denotes the approximation to $u(x_j, t_n)$, $x_j=j\,\Delta x$, $j=0,\ldots,J-1$ and $t_n=n\,\Delta t$ for positive integer n. The results are readily extended to more general finite-difference or Galerkin discretisations and more general time-steppings. It is also possible to treat systems of reaction-diffusion equations in one or more space dimensions.

2. Definitions, Assumptions and Main Theorem

In the remainder of the paper we use the following notations and norm conventions:

DEFINITION 2.1

$$\|u(\bullet,t)\|_2=\left(\int_0^1 |u(x,t)|^2\,dx\right)^{\frac{1}{2}}, \quad \|u(\bullet,t)\|_\infty=\sup_{x\in[0,1]}|u(x,t)|; \tag{2.1}$$

$$\underline{U}=[U_1,\ldots,U_{J-1}]^T; \tag{2.2}$$

$$\|\underline{U}\|_2=\left(\sum_{j=1}^{J-1}\Delta x\,|U_j|^2\right)^{\frac{1}{2}}, \quad \|\underline{U}\|_\infty=\max_{1\leq j\leq J-1}|U_j|. \tag{2.3}$$

If $v(x)$ is a function of $0\leq x\leq 1$ then we set $v_j=v(x_j)$. Similarly, if $v(t)$ is a function of $t\in[0,\infty)$ we define $v^n=v(t_n)$. □

We assume the following:

(i) $f(\bullet) \in C^2((a, b), \mathscr{R})$ for some interval $(a, b) \in \mathscr{R}$.

(ii) Equations (1.1)–(1.3) have a solution u for which the derivatives u_{xxxx} and u_{tt} exist and are uniformly bounded for $0 \leqslant x \leqslant 1$, $0 \leqslant t < \infty$, $a + \delta \leqslant u(x, t) \leqslant b - \delta$, $\forall (x, t) \in [0, 1] \times [0, \infty)$ and $\delta > 0$.

(iii) As $t \to \infty$, u approaches an equilibrium. More precisely, $\|u(\bullet, t) - \bar{u}(\bullet)\|_\infty \to 0$, where $\bar{u}$ satisfies

$$\bar{u}_{xx} + f(\bar{u}) = 0, \qquad 0 < x < 1, \tag{2.4}$$

$$\bar{u}(0) = \bar{u}(1) = 0. \tag{2.5}$$

(iv) $\bar{u}$ is an asymptotically stable equilibrium in the sense that

$$\lambda_{\max} := \max_{\phi \in H_0^1} \frac{\int_0^1 [-(\phi_x)^2 + f'(\bar{u}(x))\phi^2]\, dx}{\int_0^1 \phi^2\, dx} < 0.$$

Note that $\lambda_{\max}$ is the largest eigenvalue of the problem

$$\lambda\phi = \phi_{xx} + f'(\bar{u})\phi, \qquad 0 < x < 1, \tag{2.6}$$

$$\phi(0) = \phi(1) = 0. \tag{2.7}$$

(v) The grids are refined in such a way that $(\Delta t / \Delta x^2) \leqslant \mu < \frac{1}{2}$.

Main Theorem *Under the assumptions above, there exist constants h_0 and C, depending only upon f, u and μ, such that, for $\Delta x < h_0$ the numerical solution $\underline{U}^n = [U_1^n, \ldots, U_{J-t}^n]^T$ exists for all positive integers n and satisfies the error bound*

$$\|\underline{u}^n - \underline{U}^n\|_2 \leqslant C(\Delta t + \Delta x^2). \qquad \square$$

Note that, even though we are using an explicit scheme, the existence of the numerical solution is not guaranteed *a priori* since U_j^n may leave the domain (a, b) of definition of f. In this connection, Hoff (1978) proved that, if an invariant region S is known for the problem (1.1)–(1.3), then S is also invariant for the numerical method under the restriction

$$\Delta t \leqslant \frac{\Delta x^2}{2 + \kappa\, \Delta x^2},$$

where κ is the maximum of $|f'|$ in S.

3. Proof of main theorem

Before we start the proof of the main theorem, it is convenient to define the following:

Definition 3.1 *For each positive r we define $B_r^J = \{\underline{U} \in \mathscr{R}^{J-1} : \|\underline{U} - \bar{\underline{u}}\|_\infty \leqslant r.\}$*

Definition 3.2 *Let $e_j^n = u_j^n - U_j^n$ denote the error in the numerical method. We let T_j^n denote the local truncation error for the method:*

$$T_j^{n+1} = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\delta^2 u_j^n}{\Delta x^2} - f(u_j^n). \tag{3.1}$$

The essence of the proof lies in the following proposition, which shows that, near the equilibrium $\bar{u}$ the discretisation behaves contractively.

PROPOSITION 3.3 *Under assumptions (i)–(v) above, there exist positive constants h_0, R and β independent of Δt and Δx such that, for $\Delta x \leq h_0$, the following is true. Whenever $\underline{U}^n$ exists and $\underline{U}^n$ and $\underline{u}^n$ belong to B_R^J, then $\underline{U}^{n+1}$ is defined and*

$$\|\underline{e}^{n+1}\|_2 \leq (1 - \beta\,\Delta t)\,\|\underline{e}^n\|_2 + \Delta t\,\|\underline{T}^{n+1}\|_2. \quad \square \tag{3.2}$$

Proof. We first choose $R > 0$ sufficiently small (independently of Δx) so that the condition $\underline{U}^n \in B_R^J$ implies that each element of $\underline{U}^n$ lies in the domain of f. Subtracting (1.4) from (3.1) we obtain

$$e_j^{n+1} = e_j^n + \frac{\Delta t}{\Delta x^2}\,\delta^2 e_j^n + \Delta t[f(u_j^n) - f(U_j^n)] + \Delta t T_j^{n+1}. \tag{3.3}$$

We apply the mean value theorem twice to obtain, for $\eta_j^n \in B_R^J$

$$e_j^{n+1} = e_j^n + \frac{\Delta t}{\Delta x^2}\,\delta^2 e_j^n + \Delta t f'(\bar{u}_j) e_j^n + \Delta t f''(\eta_j^n)[(1-\gamma_j^n)(u_j^n - \bar{u}_j) + \gamma_j^n(U_j^n - \bar{u}_j)] + \Delta t T_j^{n+1}. \tag{3.4}$$

Here $\gamma_j^n \in (0, 1)$ and $\eta_j^n = (1 - s_j^n)\bar{u}_j + s_j^n[u_j^n - \gamma_j^n e_j^n]$ for $s_j^n \in (0, 1)$. Taking norms we obtain

$$\|\underline{e}^{n+1}\|_2 \leq \|I + \Delta t A\|_2\,\|\underline{e}^n\|_2 + 2K_1 R\,\Delta t\,\|\underline{e}^n\|_2 + \Delta t\,\|\underline{T}^{n+1}\|_2. \tag{3.5}$$

Here K_1 is a bound for $|f''|$ in the closed R-neighbourhood of $\bar{u}(x) : 0 < x < 1$ and A is the matrix

$$\begin{bmatrix} \theta_1 & \frac{1}{\Delta x^2} & \cdots & 0 \\ \frac{1}{\Delta x^2} & \theta_2 & \frac{1}{\Delta x^2} & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & \frac{1}{\Delta x^2} & \theta_{J-1} \end{bmatrix} \tag{3.6}$$

where $\theta_j = (-2/\Delta x^2) + f'(\bar{u}_j^n)$. Now we observe that, according to standard results, as $\Delta x \to 0$ the largest eigenvalue of A converges to the eigenvalue $\lambda_{\max}$ defined in assumption (iv)—see, for example, Kreiss (1972). Furthermore, the smallest eigenvalue can be bounded by

$$\lambda^* \geq -\frac{4}{\Delta x^2} - K_2$$

where $K_2 = \max\{f'(\bar{u}(x)) : 0 < x < 1\}$. Since A is symmetric $\|I + \Delta t A\|_2$ coincides with the spectral radius of $I + \Delta t A$. Hence, by (v), we have for Δt and Δx sufficiently small

$$\|I + \Delta t A\|_2 \leq 1 - \alpha\,\Delta t \tag{3.7}$$

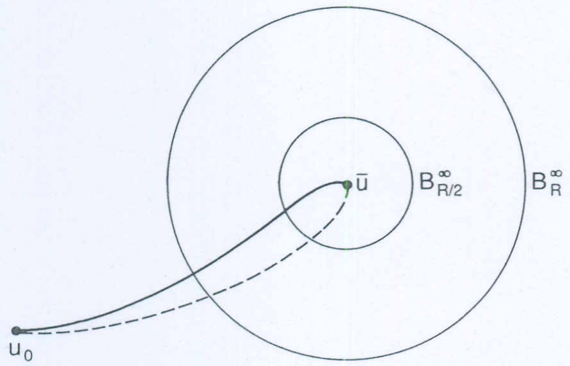

FIG. 1.

for some $\alpha > 0$. Hence (3.5) leads to the required result, perhaps after further reducing R.

Proof of Main Theorem

The idea of the proof is illustrated in Fig. 1; it is similar to the method introduced by Larsson (1989). (Note that our proof involves linearisation about the equilibrium solution whereas Larsson uses linearisation about the true time-dependent trajectory.) The true trajectory is a solid curve and the numerical trajectory a dotted curve. The ball B_R^∞ (resp. $B_{R/2}^\infty$) is a ball of radius R (resp. $R/2$) about the equilibrium $\bar{u}$ in the supremum norm. (The value of R is given by Proposition 3.3.) Let T be a time such that, for $t \geq T$, $u(\bullet, t)$ lies in $B_{R/2}$. Standard error estimates show that the conclusions of the Main Theorem hold for $0 \leq n\,\Delta t \leq T$. In particular, if $N = [T/\Delta t]$ for Δx sufficiently small, $\underline{U}^N$ will lie in B_R^J and Proposition 3.3 applies. We now show that $\underline{U}^n$ does not leave B_R^J for $n \geq N$. Let M be the largest integer for which $\underline{U}^n \in B_R^J$ for $N \leq n \leq M-1$. For Δx sufficiently small we obtain a contradiction. In fact, by iterating Proposition 3.3, we have

$$\|\underline{e}^M\|_2 \leq e^{-\beta\Delta t(M-N)}\,\|\underline{e}^N\|_2 + [1 + e^{-\beta\Delta t(M-N)}]\,\frac{T}{\beta}, \tag{3.8}$$

where T is a uniform bound for $\|\underline{T}^n\|$ with $n \geq N$ (cf. assumption (ii)). The right-hand side is $O(\Delta t + \Delta x^2)$ and, since $\underline{u}^N \in B_{R/2}^J$ and R is independent of Δx, we conclude that $\underline{U}^M \in B_R^J$, by using an inverse inequality between $\|\bullet\|_2$ and $\|\bullet\|_\infty$ and noting that $\Delta t = \mathcal{O}(\Delta x^2)$. This gives the required contradiction. The error estimate now follows by iterating Proposition 3.3 for all $n \geq N$.

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