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A NUMERICAL METHOD FOR A PARTIAL INTEGRO-DIFFERENTIAL EQUATION*

J. M. SANZ-SERNA†

Abstract. A method is considered for the integration in time of a partial integro-differential equation. The discretization technique employed is patterned after an idea of Ch. Lubich. Error bounds are derived for both smooth and nonsmooth initial data.

Key words. integro-differential equations, fractional derivative, fractional Euler method, viscoelasticity

AMS(MOS) subject classifications. 65R20, 65M10

1. Introduction. This paper is concerned with the *nonlinear partial* integro-differential equation

$$u_t + uu_x = \int_0^t (t-s)^{-1/2} u_{xx}(x,s)\,ds \tag{1.1}$$

and with the *linear* equation

$$u_t = \int_0^t (t-s)^{-1/2} u_{xx}(x,s)\,ds. \tag{1.2}$$

In both instances the real unknown function $u = u(x,t)$ is sought for $t \geqq 0$, $0 \leqq x \leqq 1$.

It is useful to compare (1.1) with the well-known Burgers equation

$$u_t + uu_x = u_{xx}. \tag{1.3}$$

In (1.3) the contribution of the *viscous* term at time t is given by $u_{xx}(x,t)$, while in (1.1) the value of the right-hand side at time t takes into account the whole history $u_{xx}(x,s)$, $0 \leqq s \leqq t$. Thus the memory integrals in (1.1)-(1.2) can be thought of as representing *viscoelastic forces*, like those present in non-Newtonian fluids [15]. In this sense, (1.1) affords a simple model equation that combines the Eulerian derivative $u_t + uu_x$ with a viscoelastic effect, just as Burgers equation provides a simple model for the study of more realistic situations involving Eulerian derivatives and viscous forces. On the other hand, it is obvious that the analysis of the linear equation (1.2) is an important step in the study of (1.1).

The problem given by (1.1) along with the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t \geqq 0 \tag{1.4}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad 0 \leqq x \leqq 1, \tag{1.5}$$

has been recently considered by Lightbourne (analytically) [7] and by Christie (numerically) [3]. The method implemented by Christie treats the (weakly singular) integral term by means of the product integration trapezoidal technique (see, e.g., [8, p. 130]). Furthermore, this author uses linear finite elements in space and employs a Crank-Nicolson time-stepping. However, the overall procedure does not achieve second order of convergence in time, due to lack of smoothness of the solution at $t = 0$ (see § 2.1 below).

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† Departamento de Matemática Aplicada y Computación, Facultad de Ciencias, Universidad de Valladolid, Valladolid, Spain.

In the present paper we employ a backward Euler method for the advancement in time of the solutions of both (1.1) and (1.2). We present a detailed analysis of the suggested method as applied to the linear problem (1.2)-(1.4)-(1.5). For smooth initial data, compatible with the boundary conditions, our estimates of the L^2-global error establish an $O(\Delta t)$ bound, uniformly in $0 \leqq t < \infty$. For (nonsmooth, incompatible) data $u_0(x) \in L^2(0,1)$ we derive an $O(\Delta t)$ error bound for t outside each layer $0 \leqq t \leqq \delta$, where the layer-width $\delta > 0$ is arbitrary. Our nonsmooth error estimates are therefore similar to those obtained by Baker, Bramble and Thomée [1] for parabolic problems.

The time-stepping technique employed in this paper was first suggested by Lubich in a very ingenious paper [10], which treats ordinary, rather than partial, integral equations. Unfortunately, the convergence result of [10] cannot be directly applied to the present situation, as it involves classical Lipschitz constants and we have to deal with the ***unbounded*** operator $u \to u_{xx}$. In other words, we find here a situation very similar to that encountered in differential equations, where classical error estimates for ordinary differential equations (ODEs) are of little use in the study of partial differential equations (see [17], [18], [19] for a detailed discussion of this point and for references on ODE stiffness-independent error estimates which avoid the use of classical Lipschitz constants). As a consequence, we have chosen to forgo the technique used in the convergence proofs of [10]. In its stead, our proofs resort to a representation of the discretization error as a *complex contour integral* (cf. [13], [14]).

A rather important point to be made is that, as discovered by Riemann and Liouville (see, e.g., [16]), the integral operator $\mathbf{I}^{1/2}$ which maps each (locally integrable) function $f(t)$, $t>0$, into the function

$$(\mathbf{I}^{1/2}f)(t) = \int_0^t (t-s)^{-1/2} f(s)\, ds$$

has the property that

$$(\mathbf{I}^{1/2}(\mathbf{I}^{1/2}f))(t) = \pi \int_0^t f(s)\, ds. \tag{1.6}$$

Thus $\sqrt{\pi}\,\mathbf{I}^{1/2}$ can be considered to be the square root of the indefinite integral operator. Other fractional powers of the latter operator may also be defined with the help of appropriate Riemann–Liouville integrals [11], [16]. It is perhaps useful to note that fractional powers of the operator $D = d/dt$ may also be defined [11], [16] and that the application of $D^{1/2}$ to both sides of (1.2) leads to the equation

$$D^{3/2}u = \sqrt{\pi}\, u_{xx}.$$

In other words the equation (1.2) is intermediate between the classical heat $Du = du_{xx}$ and wave $D^2u = c^2u_{xx}$ equations (c and d constants).

Our presentation is self-contained and does not employ either notation or results from the theory of fractional calculus or assume familiarity with Lubich's work.

The paper is organized as follows. The main results are presented in § 3. The final section contains a number of concluding remarks. Section 2 can be regarded as preparatory and is devoted to the analytical and numerical study of the linear *ordinary* integro-differential initial-value problem

$$\left(\frac{d}{dt}\right) f = -\lambda \mathbf{I}^{1/2} f, \qquad \lambda \geqq 0, \quad f(0) \text{ given}, \tag{1.7}$$

where λ denotes a given constant, $f = f(t)$, $t \geqq 0$. Here λ plays the role of a stiffness parameter (cf. with the equation $y' = -\lambda y$). Note that, if $\lambda \neq 0$, then λ can be scaled

out by appropriately choosing the units for t. In this regard, $\lambda t^{3/2}$ is a dimensionless combination.

2. Preliminaries.

2.1. Analytical results. We shall make use of Laplace transforms. If $f, g, \cdots$ are (Laplace transformable) functions defined for $0 < t < \infty$, we shall denote by capital letters $F, G, \cdots$ their respective transforms. We begin by noting that, if f and g are transformable and related by $g = \mathbf{I}^{1/2}(f)$, then the corresponding transforms satisfy

$$G(p) = (\pi/p)^{1/2} F(p). \tag{2.1}$$

This follows trivially from the rule for the transform of a convolution, because $(\pi/p)^{1/2}$ is the transform of $t^{-1/2}$. From (2.1) we conclude that the iterated application of the operator $\mathbf{I}^{1/2}$ corresponds, in the transformed realm, to multiplication by π/p. This proves (1.6) in the case of transformable f, since multiplication by $1/p$ is the transform of the integration operator.

We now turn to the study of the problem (1.7). We transform to arrive at

$$pF(p) - f(0) = -\lambda(\pi/p)^{1/2} F(p),$$

so that

$$F(p) = \frac{\sqrt{p}}{p\sqrt{p} + \lambda\sqrt{\pi}} f(0). \tag{2.2}$$

The expansion of the right-hand side of (2.2) in a negative integer power series of $\sqrt{p}$ leads to the conclusion [4, Thm. 29.2] that f can be written in the form

$$f(t) = f(0) M(\lambda\sqrt{\pi}\, t^{3/2}),$$

where M denotes the entire function

$$M(z) = 1 - (4/3)\pi^{-1/2} z + \cdots + (-1)^n \Gamma(3n/2+1)^{-1} z^n + \cdots.$$

This shows that the solution $f(t)$ of (1.7) is C^1 in $0 \leqq t < \infty$ and real analytic in $0 < t < \infty$, but, apart from the trivial case $\lambda = 0$, is *not* twice differentiable at $t = 0^+$.

For our purposes, an integral representation of f will be more useful than the previous series representation, namely, Proposition 2.1.

PROPOSITION 2.1. *The solution $f(t)$ of the initial-value problem* (1.7) *can be represented as*

$$f(t) = f(0)[R(t) - S(t)], \qquad 0 \leqq t < \infty, \tag{2.3}$$

with

$$R(t) = (2/3) \exp[\lambda^{2/3}\pi^{1/3}\omega t] + \textit{complex conjugate}, \tag{2.4}$$

$$\omega = (-1 + i\sqrt{3})/2, \tag{2.5}$$

$$S(t) = \frac{2}{3\pi} \int_0^\infty e^{-\lambda^{2/3}\pi^{1/3}\zeta^{2/3} t} \frac{d\zeta}{1+\zeta^2}. \tag{2.6}$$

Proof. For $\lambda = 0$ the result is easily checked to be true. In another case, we cut the complex p-plane along the negative real axis from $-\infty$ to 0. The transform $F(p)$ in (2.2) is then a single-valued analytic function except at the poles given by $\lambda^{2/3}\pi^{1/3}\omega$ and $\lambda^{2/3}\pi^{1/3}\omega^*$, with ω as in (2.5). (A star denotes complex conjugate.) By the inversion formula

$$f(t) = (2\pi i)^{-1} \int F(p)\, e^{pt}\, dp, \qquad t \geqq 0, \tag{2.7}$$

where the integral is taken along the imaginary axis. We now choose as a new integration path the contour obtained by juxtaposing the lower part of the cut from $-\infty$ to 0 and the upper part of the cut from 0 to $-\infty$. The residue theorem applied to (2.7), followed by some rearrangements, gives

$$f(t) = \text{residue contribution} - f(0)\pi^{-1}\int_0^\infty \frac{\lambda\sqrt{\pi}\sqrt{p}}{p^3+\lambda^2\pi}e^{-pt}\,dp. \tag{2.8}$$

A straightforward computation shows that the residue contribution is given by $f(0)R(t)$, with $R(t)$ as in (2.4). Finally, on setting in (2.8)

$$p = \zeta^{2/3}\lambda^{2/3}\pi^{1/3},$$

we arrive at (2.3). □

It is useful to note that, from (2.3),

$$|f(t)| \leqq (4/3+1/3)|f(0)| = 5/3|f(0)|, \qquad t \geqq 0, \tag{2.9}$$

which shows the continuous dependence of the solution on the datum. (A finer analysis reveals that the bound can be lowered to $|f(t)| \leqq |f(0)|$.)

Finally, let us examine the *qualitative behaviour* of $f(t)$ for $\lambda > 0$. Since $R(t) = (4/3)\exp[-(1/2)\lambda^{2/3}\pi^{1/3}t]\cos[(\sqrt{3}/2)\lambda^{2/3}\pi^{1/3}t]$, the term $f(0)R(t)$ in (2.3) represents an exponentially damped oscillation. On the other hand, $S(t)$ is clearly a decreasing function of t, with $S(0) = 1/3$. As $t \to \infty$, the asymptotic behaviour of $S(t)$ can easily be ascertained by standard means. In fact, for t large, the main contribution to the integral in (2.8) comes from $p \ll 1$. Then $p^3 + \lambda^2\pi$ can be replaced by $\lambda^2\pi$. Evaluation of the resulting integral, with the help of Euler's Γ integral, leads to

$$S(t) \sim (1/2)\lambda^{-1}\pi^{-1}t^{-3/2}, \qquad t \to \infty.$$

As a consequence we have the following.

PROPOSITION 2.2. *For* $\lambda > 0$, *the solution* $f(t)$ *of* (1.7) *possesses the asymptotic behaviour* $f(t) \sim -[f(0)/2]\lambda^{-1}\pi^{-1}t^{-3/2}$, $t \to \infty$.

2.2. The numerical method. Our discrete treatment will parallel closely the previous study of the continuous problem (1.7). With each real sequence $\{\phi_0, \phi_1, \cdots, \phi_n, \cdots\}$ we associate a *generating function*. By definition, this is the formal power series $\Phi(z) = \phi_1 z + \phi_2 z^2 + \cdots + \phi_n z^n + \cdots$, where it should be noted that ϕ_0 plays no role. It is trivial to check that the generating function of the sequence of backward differences $\{0, \phi_1 - \phi_0, \cdots, \phi_n - \phi_{n-1}, \cdots\}$ is given by

$$(1-z)\Phi(z) - z\phi_0, \tag{2.10}$$

and the generating function of the sequence of sums

$$\{0, \phi_1, \phi_2 + \phi_1, \cdots, \phi_n + \cdots + \phi_1, \cdots\}$$

is given by

$$(1-z)^{-1}\Phi(z). \tag{2.11}$$

We introduce a time-step $k > 0$ and grid-points $t_n = nk$, $n = 0, 1, 2, \cdots$. If the sequence $\{\phi_0, \phi_1, \cdots, \phi_n, \cdots\}$ consists of approximations to a function at the grid points, then the sequences $k^{-1}\{0, \phi_1 - \phi_0, \cdots, \phi_n - \phi_{n-1}, \cdots\}$ and $k\{0, \phi_1, \phi_2 + \phi_1, \cdots, \phi_n + \cdots + \phi_1, \cdots\}$ consist, respectively, of approximations to the derivative and indefinite integral at the grid points $t_n > 0$. Recalling (1.6) and (2.11), we guess that, in the generating function realm, the operator $I^{1/2}$ may be approximated by multiplication by

$$\sqrt{(\pi k)}\,(1-z)^{-1/2}. \tag{2.12}$$

The approximate method for computing the solution $f(t)$ of (1.7) is then specified by considering (2.10)–(2.12) and demanding that the generating function $\Phi(z)$ of the sequence of numerical approximation obeys the relation

$$k^{-1}[(1-z)\Phi(z) - z\phi_0] = -\sqrt{(\pi k)}\,\lambda(1-z)^{-1/2}\Phi(z), \qquad \phi_0 = f(0). \tag{2.13}$$

Equating coefficients of like powers of z, (2.13) leads to the following recursion for the computation of the numerical approximations:

$$\phi_0 = f(0),$$
$$(1+\sqrt{\pi}k^{3/2}\lambda)\phi_n = \phi_{n-1} - \sqrt{\pi}\,k^{3/2}\lambda, \tag{2.14}$$
$$\left[(1/2)\phi_{n-1} + (3/8)\phi_{n-2} + \cdots + (-1)^{n-1}\binom{-1/2}{n-1}\phi_1\right], \qquad n \geqq 1.$$

The idea behind this derivation is due to Lubich [10], [11]. Fast transform techniques can be applied to the efficient computation of the convolution sums in the right-hand sides of (2.14) [6]. A useful discussion of the advantages of Lubich's approach, as compared with product integration techniques, can be seen in the introduction of [12].

Remark 2.1. Our derivation of (2.14) has made explicit use of the relation (1.6). This is not necessary. Actually, (2.12) can be obtained directly from (2.1), by replacing p by $(1-z)/k$ [13], [14]—an alternative derivation which has the merit of being easily applicable to *arbitrary* (*Laplace transformable*) *convolution kernels*.

2.3. Some auxiliary results. From (2.13) we derive the following discrete counterpart of (2.2):

$$\Phi(z) = \frac{z\sqrt{(1-z)}}{(1-z)\sqrt{(1-z)} + \sqrt{\pi}\,\lambda k^{3/2}}f(0). \tag{2.15}$$

The discrete counterpart of Proposition 2.1 is given by the following.

PROPOSITION 2.3. *The solution* $\{\phi_n\}$ *of the recursion* (2.14) *can be represented as*

$$\phi_n = f(0)[\rho_n - \sigma_n], \qquad n = 1, 2, \cdots \tag{2.16}$$

with

$$\rho_n = (2/3)[1 - \lambda^{2/3}\pi^{1/3}\omega k]^{-n} + \textit{complex conjugate}, \tag{2.17}$$

$$\sigma_n = \frac{2}{3\pi}\int_0^\infty [1 + \lambda^{2/3}\pi^{1/3}\zeta^{2/3}k]^{-n}\frac{d\zeta}{1+\zeta^2}. \tag{2.18}$$

Proof. If $\lambda = 0$, the result is readily checked to be true. For $\lambda > 0$, the complex z-plane is cut along the real axis from 1 to ∞. Then the generating function (2.15) represents a single-valued analytic function, except at the poles given by $1 - \lambda^{2/3}\pi^{1/3}\omega k$, $1 - \lambda^{2/3}\pi^{1/3}\omega^* k$. By Cauchy's Theorem

$$\phi_n = (2\pi i)^{-1}\int \Phi(z)z^{-(n+1)}\,dz, \qquad n = 1, 2, \cdots \tag{2.19}$$

where the integral is taken along a small circle surrounding the origin. We now choose as a new integration path the contour obtained by juxtaposing the lower part of the cut from ∞ to 1 and the upper part of the cut from 1 to ∞. The residue theorem appplied to (2.19), followed by some rearrangements, yields

$$\phi_n = \text{residue contribution} - f(0)\pi^{-1}\int_1^\infty \frac{\lambda\sqrt{\pi}\,k^{3/2}\sqrt{(z-1)}}{(z-1)^3 + \lambda^2\pi k^3}z^{-n}\,dz. \tag{2.20}$$

A simple computation shows that the residue contribution is given by $f(0)p_n$, with p_n as in (2.17). The change of variables

$$z = 1 + \lambda^{2/3}\pi^{1/3}k\zeta$$

in the integral in (2.20) leads to (2.16). It should be emphasized that, as a simple computation reveals, for $n = 0$ the equality in (2.16) is not valid. In this connection note that ϕ_0 is *not* the coefficient of z^0 in the expansion of $\Phi(z)$ and hence (2.19) does not hold when $n = 1$. □

As a first consequence of the proposition we note that

$$|\phi_n| \leqq (5/3)|\phi_0|, \qquad n = 0, 1, \cdots \tag{2.21}$$

which is the discrete counterpart of (2.9). It is interesting that this stability bound is uniform both in n and in the stiffness parameter $\lambda \geqq 0$.

In the derivation of error bounds we shall need the following lemma, which we take to be well known (cf. [1]).

LEMMA. *Let α be an angle $< \pi/2$. In the complex z-plane consider the sector $S_\alpha = \{z: \arg(z) \leqq \alpha\}$. Then there exists a constant $C = C(\alpha)$ such that for each positive integer n and each z in S_α*

$$|\exp(-nz) - [1+z]^{-n}| \leqq C|z|, \tag{2.22}$$

$$|\exp(-nz) - [1+z]^{-n}| \leqq Cn^{-1}. \tag{2.33}$$

Now we are in a position to prove the following.

PROPOSITION 2.4. *If $\{\phi_n\}$ is the solution of (2.14) and $f(t)$ solves (1.7), then there exists a positive constant C (independent of k, $f(0)$, λ), such that, for each integer $n > 0$*

$$|f(t_n) - \phi_n| \leqq C\lambda^{2/3}k|f(0)|, \tag{2.24}$$

$$|f(t_n) - \phi_n| \leqq C(k/t_n)|f(0)|. \tag{2.25}$$

Proof. Subtract (2.16) from (2.3) and apply the lemma with $\alpha = \pi/3$, noting that $\zeta^{2/3}(1+\zeta^2)^{-1}$ has a finite integral over $0 \leqq \zeta < \infty$. □

Some comments are in order. The bound (2.24) is similar to those found in the B-convergence theory in numerical ODEs [5]. The stiffness parameter λ enters the bound only in as far as a larger value of λ leads to larger derivatives of the theoretical solution $f(t)$ (recall that $\lambda^{-2/3}$ provides a dimensionless unit for measuring t). However the region of large derivatives becomes narrower with increasing λ and (2.25) reveals that, outside each initial layer $0 \leqq t \leqq \delta$, $\delta > 0$, the error bound can be made totally independent of the stiffness parameter.

For fixed k and λ, the bound in (2.25) decreases like t_n^{-1} as $n \to \infty$. This is not very sharp, since by Proposition 2.2 the solution $f(t_n)$ itself decays like $t_n^{-3/2}$, if $\lambda > 0$. In order to ascertain the behaviour of the error as $n \to \infty$, for fixed k and λ, we observe that an asymptotic estimation for ϕ_n can be obtained in (2.20) by discarding the exponentially small residue contribution and replacing $(z-1)^3 + \lambda^2\pi k^3$ by $\lambda^2\pi k^3$. A change of variables reduces the resulting integral to Euler's B integral. This gives ($\lambda > 0$)

$$\phi_n \sim -f(0)\lambda^{-1}\pi^{-3/2}k^{-3/2}\mathrm{B}(3/2, n-3/2), \qquad n \to \infty.$$

On expressing the B function in terms of Γ functions and employing Stirling's asymptotic approximation to Γ, it is possible to write

$$\phi_n \sim -(f(0)/2)\lambda^{-1}\pi^{-1}(kn)^{-3/2}, \qquad n \to \infty.$$

Comparison with Proposition 2.2 leads finally to the conclusion:

$$\phi_n \sim f(t_n), \qquad n \to \infty. \tag{2.26}$$

Thus the *relative error* tends to 0 as n increases with k fixed, a rather surprising behaviour that does not occur in the approximation of the equation $dy/dt = -\lambda y$, $\lambda > 0$, by means of standard A-stable Runge-Kutta or multistep methods. However, note that, as distinct from (2.25), the relation (2.26) is not uniform in λ.

3. Main results. The linear problem given by (1.2)-(1.4)-(1.5) will be considered next. We assume that the initial datum u_0 is in $X = L^2(0, 1)$. Denoting by A the (unbounded) operator in X given by $u \to u_{xx}$ with boundary conditions (1.4), we can rewrite our problem in the abstract form:

$$u_t = \mathrm{I}^{1/2}(Au), \quad t \geqq 0, \quad u(0) = u_0 \in X. \tag{3.1}$$

This is similar to (1.7), with A playing the role of $-\lambda$. The application of the time-stepping recursion (2.14) in the present circumstances leads to the following formulae, where $U_n \in X$ denotes the approximation to $y(t_n)$:

$$U_0 = u(0),$$

$$(1 - \sqrt{\pi}\,k^{3/2}A)U_n = U_{n-1} + \sqrt{\pi}\,k^{3/2}A, \tag{3.2}$$
$$\left[(1/2)U_{n-1} + (3/8)U_{n-2} + \cdots + (-1)^{n-1}\binom{-1/2}{n-1}U_1\right], \qquad n \geqq 1.$$

Note that U_n is well defined, since $(1 - \sqrt{\pi}\,k^{3/2}A)^{-1}$ is bounded operator defined in the whole of X and a simple induction argument proves that the element in square brackets belongs to the domain of A.

Let $-\lambda_m$, $m = 1, 2, \cdots$, be the eigenvalues of A, written in increasing order of magnitude, and let w_m be the corresponding X-orthogonal eigenfunctions, so that A possesses the spectral representation

$$Av = -\sum_m \lambda_m(v, w_m)w_m, \qquad \lambda_m \geqq 0. \tag{3.3}$$

Here $(\ ,\ )$ denotes the inner product in X. We also need the spaces Y_s, $s \geqq 0$, defined as follows. An element v in X belongs to Y_s if and only if

$$\|v\|_s = \left(\sum_m \lambda_m^s(v, w_m)^2\right)^{1/2} < \infty. \tag{3.4}$$

Therefore $Y_0 = X$ and $\|\ \|_0$ denotes the norm in X. Our main results are as follows.

THEOREM 3.1. (Stability.) *Assume that U_0 belongs to Y_s, $s \geqq 0$. If $\{U_n\}$ denotes the solution of the recursion (3.1) then, for $n = 0, 1, 2, \cdots$*

$$\|U_n\|_s \leqq (5/3)\|U_0\|_s. \tag{3.5}$$

Proof. Use the spectral decomposition (3.3) and apply (2.21). □

THEOREM 3.2. (Error bounds for nonsmooth initial datum.) *Let $u(t)$, $\{U_n\}$ denote the solutions of (3.1), (3.2), respectively. Then there exists a constant C (independent of k and u_0) such that for $n = 1, 2, \cdots$*

$$\|u(t_n) - U_n\|_0 \leqq C(k/t_n)\|u_0\|_0. \tag{3.6}$$

Proof. Use the spectral decomposition (3.3) and apply (2.25). □

THEOREM 3.3. (Error bounds for smooth initial datum.) *With the notation of the previous theorem, assume that the initial datum u_0 belongs to $Y_{4/3}$. Then there exists a* *constant C, independent of k and u_0, such that for $n = 0, 1, 2, \cdots$*

$$\|u(t_n) - U_n\|_0 \leqq Ck\|u_0\|_{4/3}. \tag{3.7}$$

Proof. Use the spectral decomposition (3.3) and apply (2.24). □

Remark 3.1. Since the eigenvalues and eigenfunctions of the operator $u \to u_{xx}$ with boundary conditions (1.4) are given by $-(2\pi m)^2$ and $(\sqrt{2}/2)\sin[2\pi mx]$, respectively, it is evident that the expansion of a function in series of eigenfunctions w_m is identical to its sine-Fourier series. Furthermore if $u_0 \in Y_{4/3}$, then the series $\sum_m m^{8/3}(u_0, w_m)^2$ converges and therefore $|(u_0, w_m)| = o(m^{-4/3})$. The Weierstrass *M*-criterion shows that the sine-Fourier series for v converges uniformly in x, so that in particular u_0 is continuous and satisfies the boundary conditions (1.4). Thus, the last theorem refers to a situation of *compatible* initial data.

Remark 3.2. It is clear that error estimates similar to (3.6) (respectively, (3.7)) can be obtained for the higher norms of the error $\|u(t_n) - U_n\|_s$, $s \geqq 0$, provided that the initial datum lies in Y_s (respectively, $Y_{s+4/3}$).

4. Concluding remarks. (i) The only property of the operator A used in the proof of the Theorems 3.1, 3.2 and 3.3 is the existence of the spectral decomposition (3.3). Therefore all the results in § 3, with the exception of those under Remark 3.1, are valid for any operator A in a Hilbert space X for which (3.3) holds. This includes linear elliptic operators $\sum_{ij} \partial_i(a_{ij}\partial_j u) - a_0 u$, $a_0 \geqq 0$, in smooth bounded domains Ω of R^d, $d = 1, 2, 3$, with constant or smooth variable coefficients a_{ij}, a_0 and homogeneous Dirichlet, homogeneous Neumann or (if Ω is a parallelepiped) periodic boundary conditions ($X = L^2(\Omega)$).

(ii) More generally, a method similar to that presented in the paper can be constructed for the equation

$$Du = R^\beta Au, \tag{4.1}$$

with $D = d/dt$, β a given constant $\beta > 0$ and R^β the Riemann-Liouville operator with kernel $(t-s)^{\beta-1}$. ($\beta = \frac{1}{2}$ corresponds of course to the case treated so far in the paper.) An analysis parallel to that carried out in the paper holds if $0 < \beta < 1$, leading to an $O(\Delta t)$ bound for the X-norm of the global error, uniformly in $0 \leqq t < \infty$, for data u_0 in X. In the $u_0 \in Y_{2/(1+\beta)}$ and to an $O(\Delta t)$ bound outside any initial layer, for data u_0 in X. In the derivation of those bounds, use must be made of the lemma with an angle $\alpha = (\pi\beta)/(1+\beta)$. Since the condition $\alpha < \pi/2$ is necessary for the result in the lemma to hold, our analysis cannot be carried out for $\beta \geqq 1$. In fact, for $\beta \geqq 1$ convergence bounds that hold uniformly for large t cannot exist. This is easily seen by noting that, for $\beta = 1$, differentiation of (4.1) with respect to t leads to $u_{tt} = Au$, a wave equation.

(iii) For numerical purposes the "elliptic" operator A in (3.2) must be replaced by a finite-dimensional approximation A_h, by means of a finite-element, a finite-difference or a spectral technique. The analysis of those fully discrete formulations of our algorithm can be performed by combining our results with standard error bounds for elliptic problems (cf. the techniques in [1]).

(iv) The method (3.2) can be readily modified to accommodate the nonlinear equation (1.1). It is sufficient to add to the left-hand side the contribution of the nonlinear term at time t_n. In principle, the analysis of the resulting nonlinear discretization can be carried out employing the techniques of [9].

(v) Camino [2] has given numerical results corresponding to the problem (1.7) integrated according to (2.14) and to the problem (1.1)-(1.4)-(1.5) integrated according to the procedure in (iv), with piecewise linear finite-elements for the space discretization. His results are in perfect agreement with the present analysis.

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