ON THE USE OF THE PRODUCT APPROXIMATION TECHNIQUE IN NONLINEAR GALERKIN METHODS

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The term 'product approximation' (PA) was introduced by Christie *et al.*[1] to refer to a technique for the treatment of nonlinear terms in the finite element method. This technique is also referred to as 'group formulation'.[2] The present note contains a discussion of some of the practical advantages and disadvantages of the PA procedure.

For simplicity, we restrict ourselves to the model elliptic equation

$$\Delta u + bu_x + f(u) = 0 \tag{1}$$

in a bounded region in R^n, with suitable boundary conditions. In (1), Δ denotes the Laplacian operator in n space variables $x, y, \ldots$ and b is a constant. The numerical solution of many time-dependent problems requires an equation similar to (1) to be solved at each time step.

When Galerkin's method with shape functions ϕ_j, $j = 1, \ldots, N$ is used, the nonlinear term in (1) contributes to the discrete system of nonlinear equations with an N-vector $\mathbf{F}$ whose ith entry is given by the inner product

$$F_i = (f(\sum \phi_j U_j), \phi_i) \tag{2}$$

Here, U_j, $j = 1, \ldots, N$ denote the nodal parameters of the numerical solution $U = \sum \phi_j U_j$. (For simplicity, we assume the elements to be of Lagrange type. Hermite elements can be treated in a similar manner.) The vector $\mathbf{F}$ must be evaluated (at least) once within each step of the iterative method being employed for the solution of the discrete nonlinear system. Typically, this evaluation requires the use of numerical quadrature to compute the inner products (2).

In the PA technique, (2) is replaced by

$$F_i = \sum_j (\phi_i, \phi_j) f(U_j) \tag{3}$$

a device which does not imply loss of accuracy as shown both theoretically and experimentally in References 1, 3 and 4. In (3), the inner products (ϕ_i, ϕ_j) can be computed once and for all before the iteration starts. From here, it is sometimes concluded[1,5] that the PA improves on standard quadrature Galerkin methods (QG) as far as *computational effort* is concerned. In our experience this is probably true in one-dimensional problems ($n = 1$), but is not necessarily so in higher dimensions. In order to clarify this the following remarks are in order.

1. In one-dimensional problems the following implementation of the PA technique has proved to be highly useful (see Reference 6 among many others). The mass matrix M with entries (ϕ_i, ϕ_j) is assembled before the iteration begins and stored in band form. The bandwidth is usually a small integer m. Then, $\mathbf{F}$ is evaluated at each step as the product $M\mathbf{f}$, where $\mathbf{f}$ is the vector whose ith entry equals $f(U_i)$. Thus, the evaluation of $\mathbf{F}$ requires Nm multiplications and N evaluations of f, and is easily coded. This compares favourably with QG methods (even

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more so when higher order elements are used, as these would require expensive quadrature rules if a QG method were employed).

2. In two or three space dimensions the implementation described above for the PA technique is very inefficient.[4] In fact, for a uniform rectangular or triangular mesh in the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, with element diameter h, there are $O(h^{-2})$ nodes or entries in $\mathbf{F}$, while the bandwidth of M is $O(h^{-1})$, so that the procedure in remark 1 above, for the computation of $M\mathbf{f}$, leads to a total of $O(h^{-3})$ flops. This compares unfavourably with the $O(h^{-2})$ count in QG methods. Efficient implementations of the PA technique will, in general, resort to the computation of $\mathbf{F}$ by assembly within each iteration. The procedure is then not substantially different from QG algorithms and so the advantages of the PA technique, if any, will not be marked. A final assessment will depend critically on the particular implementation, the type of element and the problem at hand. Note that (3) can be evaluated analytically for any function f, as distinct from the situation for (2). In favourable geometries, it might be useful to resort to analytical evaluation.

3. When PA is used, the Jacobian $\partial\mathbf{F}/\partial\mathbf{U}$ with entries $(\phi_i, \phi_j)\, f'(U_j)$ is readily computed, but unfortunately is not symmetric[3,4]. Thus, if Jacobians are required (as they will be in Newton's method) the PA technique may introduce nonsymmetric matrices in situations which otherwise demand only symmetric ones (e.g. equation (1) when $b = 0$). This doubles the storage requirements and greatly increases the computational cost. In these instances, the PA technique is not to be favoured, except perhaps in one-dimensional problems.

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