# Barrelledness conditions on $C_0(E)$

By

A. Marquina and J. M. Sanz Serna

**Abstract.** Some conditions of barrelledness are considered and studied on the space $C_0(E)$, defined as follows: If $E$ is a real or complex Hausdorff locally convex space and $\mathscr{P}_E$ is a saturated family of seminorms, defining the original topology of $E$, then the vector space of all the sequences $\bar{f} = \{\bar{f}(n) : n \in \mathbb{N}\}$ in $E$, convergent to zero, provided with the locally convex topology

$$\bar{p}(\bar{f}) = \sup\{p(\bar{f}(n)) : n \in \mathbb{N}\} \qquad p \in \mathscr{P}_E$$

is defined as the space $C_0(E)$. The main result of the paper is the following characterization: $C_0(E)$ is quasibarrelled (see [3], p. 367) if and only if, $E$ is quasibarrelled and the strong dual of $E$ has property (B) (see [5], p. 30, for definition). We obtain, as a consequence, commutativity properties of the operator $C_0$, acting on certain inductive limits (3.3 Theorem). We also prove that $C_0$ does not commute with uncountably strict inductive limits. In particular, there are ultrabornological spaces $E$ for which $C_0(E)$ is not quasibarrelled. 3.1. Example provides a complete $\varepsilon$-tensor product of two complete ultrabornological spaces which is not quasibarrelled.

**1. Notation and introduction.** The vector spaces used here will be defined over the field of the real or complex numbers. With the expression "locally convex space" we shall mean a Hausdorff topological vector space with a zero neighbourhood basis consisting of convex sets. If $\langle E, E' \rangle$ is a dual pair (see [3]), we shall denote by $\sigma(E, E')$, $\mu(E, E')$ and $\beta(E, E')$, the weak, the Mackey and the strong topology on $E$. If $E$ is a locally convex space, then $E'$ will denote the topological dual of $E$. If $U$ is an absolutely convex set in $E$, then $p_U$ will represent the Minkowski functional of $U$ defined on the linear hull of $U$, $E_U$.

**1.1. Definition.** If $E$ is a locally convex space and $\mathscr{P}_E$ is a saturated family of seminorms defining the original topology of $E$, then $C_0(E)$ is defined as the space of all null sequences in $E$, provided with the locally convex topology generated by the system of seminorms

$$\bar{p}(\bar{f}) = \sup\{p(\bar{f}(n)) : n \in \mathbb{N}\}, \qquad p \in \mathscr{P}_E$$

for $\bar{f} = \{\bar{f}(n) : n \in \mathbb{N}\}$, convergent to zero in $E$.

The space $C_0(E)$ has been studied in various situations. In [6], the topological dual of $C_0(E)$, was characterized improving a result given in [1]. Perhaps one of the major problems about the space $C_0(E)$ is to know whether $C_0(E)$ is a barrelled, quasibarrelled, ..., space provided $E$ satisfies the same property. This problem is closely related with the question of commutativity of the topological operator $C_0$ acting on certain inductive limits and it was treated in [2]. In this paper, we give a complete characterization of the spaces $C_0(E)$ that are quasibarrelled, giving a partial answer to the above question. As consequence, the barrelledness on $C_0(E)$ is characterized in a very general situation. 3.3. Theorem includes, as a particular case, the result of [2] and 3.1. Example provides an example of a complete ultrabornological space $E$ such that $C_0(E)$ is not quasibarrelled.

**1.2. Definition.** Let $E$ be a locally convex space. Let $\mathfrak{U}$ be a saturated family (in the sense of [3], § 21) of closed bounded absolutely convex subsets of $E$. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence in $E$. We say that $\{f_n : n \in \mathbb{N}\}$ is a $\mathfrak{U}$ totally summable sequence, if there is an element $A \in \mathfrak{U}$, such that

$$\sum_{\mathbb{N}} p_A(f_n) < +\infty$$

i.e., $\{p_A(f_n) : n \in \mathbb{N}\} \in \ell^1$.

If $\mathscr{B}(E)$ is the family of all closed bounded absolutely convex subsets of $E$, then the above definition gives the concept of a totally summable sequence (see [5], p. 29).

**1.3. Definition.** Following Pietsch ([5], p. 23), a sequence $\{u_n : n \in \mathbb{N}\}$ of $E'$, the topological dual of a locally convex space $E$, is $\sigma(E', E)$-summable if for every $f \in E$, the condition

$$\sum_{\mathbb{N}} |\langle f, u_n\rangle| < +\infty$$

is satisfied. We shall denote by $\varphi\langle E'\rangle$ the vector space of all sequences in $E'$ which are $\sigma(E', E)$-summable and generate finite-dimensional subspaces of $E'$.

Let $E$ be a locally convex space. Let $\mathfrak{U}$ be a saturated family of closed bounded absolutely convex subsets of $E$. We shall denote by $\ell^1\langle\mathfrak{U}\rangle$ the set of all $\mathfrak{U}$-totally summable sequences of $E$, which is a vector space with the pointwise operations. The vector space of all totally summable sequences of $E$ will be denoted by $\ell^1\langle E\rangle$. The family of all $\sigma(E', E)$ closed absolutely convex and equicontinuous subsets of $E'$ will be denoted in the sequel by $\mathscr{U}(E')$.

The following result was given in [6] and ([9], p. 463/464, 18a) and b)):

**1.4. Proposition.** *Let $E$ be a locally convex space. Then the topological dual of $C_0(E)$ is algebraically isomorphic to $\ell^1\langle\mathscr{U}(E')\rangle$.*

1.5. Remark. The equicontinuous sets on $C_0(E)$ are characterized as follows: A subset $H \subset C_0(E)'$ (the topological dual of $C_0(E)$) is equicontinuous if and only if there is an equicontinuous set $U \in \mathcal{U}(E')$ and a constant $M > 0$, such that

$$\sum_{\mathbb{N}} p_U(\bar{u}(n)) \leqq M$$

for all $\bar{u} \in H$. This remark was settled in [6].

Following Pietsch, we shall denote by $l_\pi^1\{E\}$ the locally convex space of all absolutely summable sequences of the locally convex space $E$, provided with the $\pi$-topology.

**1.6. Proposition.** *Let $E$ be a locally convex space. The topological dual of $C_0(E)$ is algebraically isomorphic to a sequentially dense subspace of $\ell_\pi^1\{E'[\beta(E', E)]\}$.*

Proof. If $\{u_n : n \in \mathbb{N}\} \in \ell_\pi^1\{E'[\beta(', E)]\}$, then for every positive integer $k$, we define the sequence $\{u_n^{(k)} : n \in \mathbb{N}\}$ as follows: $u_n^{(k)} = u_n$ if $1 \leqq n \leqq k$, and $u_n^{(k)} = 0$ if $n > k$. Thus, for $k \geqq 1$, $\{u_n^{(k)} : n \in \mathbb{N}\} \in \varphi\langle E'\rangle$. On the other hand, if $W$ is a closed absolutely convex neighbourhood of zero in $E'[\beta(E', E)]$, we have that

$$\sum_{\mathbb{N}} p_W(u_n) < +\infty .$$

Let us take $\varepsilon > 0$ arbitrary, then there is a positive integer $n_0$ such that

$$\sum_{n \geqq n_0} p_W(u_n) < \varepsilon$$

and hence

$$\pi_W[\{u_n : n \in \mathbb{N}\} - \{u_n^{(k)} : n \in \mathbb{N}\}] = \sum_{n > k} p_W(u_n) < \varepsilon \quad \text{for all} \quad k \geqq n_0,$$

(using the notation of [5] for the seminorms of the $\pi$-topology). Thus,

$$\pi - \lim_{k \to +\infty} \{u_n^{(k)} : n \in \mathbb{N}\} = \{u_n : n \in \mathbb{N}\}.$$

It follows that $\varphi\langle E'\rangle$ will be sequentially dense in $\ell_\pi^1\{E'[\beta(E', E)]\}$. Since

$$\varphi\langle E'\rangle \subset \ell^1\langle \mathcal{U}(E')\rangle \subset \ell_\pi^1\{E'[\beta(E', E)]\}$$

we have the result by 1.4. Proposition. Q.e.d.

## 2. The Main Result.

**2.1. Proposition.** *Let $E$ be a locally convex space. Then, the topology induced by the $\pi$-topology of $\ell_\pi^1\{E'[\beta(E', E)]\}$ on $C_0(E')$, coincides with $\beta[C_0(E)', C_0(E)]$.*

Proof. If $B$ is a closed bounded absolutely convex subset of $E$, then it is easy to see that

$$\sup\{|\langle \bar{f}, \bar{u}\rangle| : \bar{f} = \{\bar{f}(n) : n \in \mathbb{N}\} \in C_0(E) \quad \text{and} \quad \bar{f}(n) \in B, \forall n\} = \\ = \sum_{\mathbb{N}} \sup\{|\langle g, \bar{u}(n)\rangle| : g \in B\}$$

where $\bar{u} = \{\bar{u}(n) : n \in \mathbb{N}\} \in C_0(E)'$, (see Proposition 1 of [7]). From this remark and the fact that the family of subsets of $C_0(E)$:

$$B^* = \{\bar{f} : \bar{f} \in C_0(E) \quad \text{and} \quad \bar{f}(n) \in B, \forall n\},$$

$B$ running through the family of all closed bounded absolutely convex subsets of $E$, is a fundamental system of bounded subsets in the locally convex space $C_0(E)$, we obtain: The family of seminorms on $C_0(E)'$,

$$(2.1.1) \qquad \bar{u} \to \sum_{\mathbb{N}} p_{B^0}(\bar{u}(n))$$

where $B^0 = \{v : v \in E', |v(f)| \leqq 1, \text{ for all } f \in B\}$, and $B$ runs through the family $\mathscr{B}(E)$ of all bounded closed absolutely convex subsets of $E$, defines the topology $\beta[C_0(E)', C_0(E)]$. On the other hand, the seminorms (2.1.1) define exactly the $\pi$-topology on $\ell^1\{E'[\beta(E', E)]\}$. Q.e.d.

**2.2. Remark.** The normed spaces are examples of locally convex spaces $E$, for which the equality

$$C_0(E)' = \ell^1\{E'[\beta(E', E)]\}$$

is satisfied. The following remark shows that the inclusion

$$(2.2.1) \qquad C_0(E)' \subset \ell^1\{E'[\beta(E', E)]\}$$

may be strict. On the one hand, for every locally convex space $E$ one has:

$$C_0(E[\sigma(E, E')])' = \varphi\langle E'\rangle.$$

Let $E$ be a locally convex space such that $E[\sigma(E, E')]$ is not quasibarrelled. Then there exists an infinite-dimensional bounded sequence $\{u_n : n \in \mathbb{N}\}$ in $E'[\beta(E', E)]$, whence $\{2^{-n} u_n : n \in \mathbb{N}\} \in \ell^1\langle E'[\beta(E', E)]\rangle \sim \varphi\langle E'\rangle$. Thus, for the locally convex space $F = E[\sigma(E, E')]$ the above inclusion is strict.

**2.3. Example.** The following example is stronger than the one discussed in 2.2. Remark in the sense that it shows that the inclusion (2.2.1) may be strict, for locally convex spaces $E$, such that the original topology of $E$ coincides with the Mackey topology $\mu(E, E')$ (the so called Mackey spaces).

In Köthe's book ([3], § 27, p. 369) an example is given of a vector subspace $F_0$ of $\ell^1$, dense and different from $\ell^1$, such that with the induced norm of $\ell^1$ is a barrelled space. The strong dual (or conjugate) of $F_0$ will be $\ell^\infty$ (i.e. the space of all the bounded sequences of scalars with the supremum norm topology). We set $E = \ell^\infty[\mu(\ell^\infty, F_0)]$. Since $F_0 \neq \ell^1$, then there is an element $f \in \ell^1 \sim F_0$. Since $\ell^1$ is the completion of the subspace $F_0$, there exists a sequence $\{f_n : n \in \mathbb{N}\}$ in $F_0$, which is absolutely summable in $\ell^1$, i.e. $\sum_{\mathbb{N}} \|f_n\|_1 < +\infty$ (where $\|\cdot\|_1$, is the $\ell^1$-norm) and

$$f = \sum_{\mathbb{N}} f_n, \quad \text{in } \ell^1$$

(for a proof of this fact see for instance [5], p. 55, 3.2.2. Lemma). We set

$$s_k = \sum_{n=1}^{k} f_n, \qquad k \geqq 1.$$

Let $\bar{g} = \{f_n : n \in \mathbb{N}\}$. Since $F_0$ is barrelled, the strong topology $\beta(F_0, \ell^\infty)$ coincides with the induced topology by the norm of $\ell^1$, and therefore $\bar{g} \in \ell^1\{F_0[\beta(F_0, \ell^\infty)]\}$, since $\bar{g}$ is $\|\cdot\|_1$-summable. On the other hand, $\bar{g} \notin \ell^1\langle\mathcal{U}(E')\rangle$. Indeed, let us suppose that $\bar{g} \in \ell^1\langle\mathcal{U}(E')\rangle$, then, for a suitable $U \in \mathcal{U}(E')$, we have that

$$M = \sum_{\mathbb{N}} p_U(f_n) < +\infty .$$

Thus, for every $k \in \mathbb{N}$,

$$p_U(s_k) = p_U\left(\sum_{n=1}^{k} f_n\right) \leqq \sum_{n=1}^{k} p_U(f_n) \leqq M .$$

We set $V = \{u : u = M \cdot v;\ v \in U\}$. Then, $V \in \mathcal{U}(E')$ and $s_k \in V$, for all $k \in \mathbb{N}$. Since $V$ is $\sigma(E', E)$-compact and absolutely convex, it follows that $\{s_k : k \in \mathbb{N}\}$ has a $\sigma(E', E)$-adherent point $f_0 \in V \subset E' = F_0$. On the other hand, $\{s_k : k \in \mathbb{N}\}$ converges to $f$ in $(\ell^1, \|\cdot\|_1)$, hence in $\ell^1[\sigma(\ell^1, \ell^\infty)]$; since $f_0$ is in particular an adherent point of $\{s_k : k \in \mathbb{N}\}$ in the space $\ell^1[\sigma(\ell^1, \ell^\infty)]$, one obtains $f = f_0$, and, therefore, $f$ lies in $F_0$, and this is a contradiction.

**2.4. Remark.** The result of 2.3. Example is valid for any normed barrelled space $X$ and any barrelled dense subspace $F_0 \subset X$, and different from $X$. (Use the same proof!)

**2.5. Definition.** Let $E$ be a locally convex space. Following A. Pietsch ([5], p. 30), $E$ has *property* (B), if for every bounded subset $H$ of $\ell^1_\pi\{E\}$, there is a closed bounded absolutely convex set $B$ in $E$ such that

$$\sum_{\mathbb{N}} p_B(f_n) \leqq 1 , \quad \text{for all} \quad \{f_n : n \in \mathbb{N}\} \in H .$$

Now, we can give our main result:

**2.6. Theorem.** *Let $E$ be a locally convex space. The following properties are equivalent:*

(2.6.1) *$E$ is quasibarrelled and $E'[\beta(E', E)]$ has property* (B).

(2.6.2) *$C_0(E)$ is quasibarrelled.*

*If $C_0(E)$ is quasibarrelled, then,*

$$C_0(E)' = \ell^1\{E'[\beta(E', E)]\} .$$

Proof. (2.6.1) $\rightarrow$ (2.6.2): If $E$ is quasibarrelled, then, every $\beta(E', E)$-bounded subset of $E'$ is equicontinuous, and therefore we have that

$$\ell^1\langle\mathcal{U}(E')\rangle = \ell^1\langle E'[\beta(E', E)]\rangle .$$

Since $E'[\beta(E', E)]$ has property (B) by hypothesis, applying 1.5.6. Proposition from ([5], p. 30), we obtain that

$$\ell^1\langle E'[\beta(E', E)]\rangle = \ell^1\{E'[\beta(E', E)]\} ,$$

and therefore $C_0(E)' = \ell^1\{E'[\beta(E', E)]\}$. Let us see now that $C_0(E)$ is quasibarrelled. Let $H$ be a bounded subset of the strong dual of $C_0(E)$. From 2.1. Proposition it results that $H$ is $\pi$-bounded. Since the strong dual of $E$ has property (B), there is a strongly bounded subset $B$ of $E'$, which can be chosen absolutely convex and $\sigma(E', E)$-closed (since the $\sigma(E', E)$-closure of a $\beta(E', E)$-bounded subset of $E'$ is $\beta(E', E)$-bounded), and such that for all $\bar{u} \in H$, the inequality

$$\sum_{\mathbb{N}} p_B(\bar{u}(n)) \leqq 1$$

is satisfied.

$E$ being quasibarrelled, $B$ is equicontinuous, therefore $B \in \mathcal{U}(E')$, since $B$ is $\sigma(E', E)$-closed. By an appeal to 1.5. Remark it results that $H$ is an equicontinuous subset of $C_0(E)'$.

(2.6.2) $\to$ (2.6.1): Let us suppose that $C_0(E)$ is quasibarrelled. We define the linear mapping $p_1$ from $C_0(E)$ onto $E$, as follows: if $\bar{f} \in C_0(E)$, then $p_1(\bar{f}) = \bar{f}(1)$, the first component of the sequence $\bar{f}$. $p_1$ is clearly continuous. On the other hand, if $W$ is a neighbourhood of zero in $C_0(E)$, there is a continuous seminorm $p$ such that for a certain $\varepsilon > 0$,

$$V_p^* = \{\bar{f} \in C_0(E);\ \bar{p}(\bar{f}) = \sup\{p(\bar{f}(n)) : n \in \mathbb{N}\} < \varepsilon\}$$

is a subset of $W$. Then, $U_p = \{f : f \in E;\ p(f) < \varepsilon\}$ is a neighbourhood of zero in $E$, such that

$$p_1(W) \supset p_1(V_p^*) \supset U_p$$

because if $f \in U_p$, we define $\bar{f} = \{\bar{f}(n) : n \in \mathbb{N}\}$, such that $\bar{f}(1) = f$ and $\bar{f}(n) = 0$ if $n > 1$, and we have that $\bar{f} \in V_p^*$ and $p_1(\bar{f}) = f$. Thus, $p_1$ is open, and therefore it is a quotient mapping. Thus, $E$ as a quotient of $C_0(E)$, will be quasibarrelled. Combining 1.6. Proposition, 2.1. Proposition and the fact that $C_0(E)'$ is quasi-complete for the topology $\beta[C_0(E)', C_0(E)]$ ([3], § 23), we can deduce that

$$C_0(E)' = \ell^1\{E'[\beta(E', E)]\}.$$

To show that $E'[\beta(E', E)]$ has property (B) the following argument works: If $H \subset \ell^1\{E'[\beta(E', E)]\}$ is $\pi$-bounded, $C_0(E)$ being quasibarrelled, 2.1. Proposition yields that $H$ is equicontinuous; from 1.5. Remark there is an equicontinuous subset $U$ of $E'$, $U \in \mathcal{U}(E')$ and a constant $M > 0$ such that for all $\bar{u} \in H$,

$$\sum_{\mathbb{N}} p_U(\bar{u}(n)) \leqq M.$$

We set $B = \{v : v = M \cdot u \text{ with } u \in U\}$. Therefore, from the Banach-Mackey theorem ([3], § 20), $B$ will be $\beta(E', E)$-bounded and such that

$$\sum_{\mathbb{N}} p_B(\bar{u}(n)) \leqq 1, \quad \text{for all} \quad \bar{u} \in H.$$

Thus, $E'[\beta(E', E)]$ has property (B). Q.e.d.

**2.7. Remark.** From 2.6. Theorem it follows that if $E$ is a quasibarrelled DF-space, then $C_0(E)$ is a quasibarrelled DF-space, since $E'[\beta(E', E)]$ is metric and every metric locally convex space has property (B) by ([5], 1.5.8., p. 31). In particular, if $E$ is an LB-space, then, $C_0(E)$ is quasibarrelled.

### 3. Concluding Remarks.

**3.1. Example.** Let $I$ be an uncountable index set. Let $E = \mathbb{K}^{(I)}$ be the locally convex direct sum of spaces $E_i = \mathbb{K}$ (being $\mathbb{K}$ the field of scalars), $i \in I$. In ([5], 1.5.7. Example) it is proved that $E'[\beta(E', E)]$ does not satisfy property (B). Since $E$ is the inductive limit of finite-dimensional subspaces, it follows that $E$ will be barreled (even, "ultrabornological", i.e. inductive limit of Banach spaces). If we apply our 2.6. Theorem, we obtain that $C_0(E)$ is not quasibarrelled, and, therefore, it is not the inductive limit of the corresponding subspaces $C_0(F)$, $F$ running through the finite-dimensional subspaces of $E$. Thus, $C_0$ does not commute with the strict uncountable inductive limits. The space $C_0(E)$ can be interpreted as a complete $\varepsilon$-tensor product (see [5], p. 108) of two complete ultrabornological spaces, such that it is not quasibarrelled.

Recall that a locally convex space $E$ is *locally complete* if for every closed bounded absolutely convex set in $E$, $B$, then, the normed space $E_B$ is Banach. A Mackey-Cauchy (resp. Mackey-convergent) sequence in $E$, is a Cauchy (resp. convergent) sequence in certain $E_B$, $B$ being a closed bounded absolutely convex set in $E$.

**3.2. Corollary.** *Let $E$ be a locally complete locally convex space. Then the following conditions are equivalent:*

(3.2.1) *$E$ is barrelled and $E'[\beta(E', E)]$ has property* (B).

(3.2.2) *$C_0(E)$ is barrelled.*

Proof. If $\{(f_n^{(m)} : n \in \mathbb{N}) : m \in \mathbb{N}\}$ is a Mackey-Cauchy sequence in $C_0(E)$, then $\{f_n^{(m)} : m \in \mathbb{N}\}$ is a Mackey-Cauchy sequence in $E$, hence convergent to some $f_n \in E$, $(n \in \mathbb{N})$; it follows easily that $\{f_n : n \in \mathbb{N}\} \in C_0(E)$ and that $\{(f_n^{(m)} : n \in \mathbb{N}) : m \in \mathbb{N}\}$ converges to $\{f_n : n \in \mathbb{N}\}$. Thus, $C_0(E)$ is locally complete if (and only if) $E$ is locally complete. Applying 2.6. Theorem we have the result. Q.e.d.

**3.3. Theorem.** *Let $E$ be a locally complete locally convex space. Let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of locally complete subspaces of $E$ such that*

$$E = \bigcup \{E_n : n \in \mathbb{N}\}.$$

*If $C_0(E)$ is barrelled, then,*

(3.3.1) $C_0(E) = \bigcup \{C_0(E_n) : n \in \mathbb{N}\}$.

(3.3.2) *$C_0(E)$ is the inductive limit of the sequence $\{C_0(E_n) : n \in \mathbb{N}\}$.*

Proof. (3.3.1): If $f \in C_0(E)$, then, $\{f(n) : n \in \mathbb{N}\}$ is a null sequence in $E$; since $E$ is locally complete, the closed absolutely convex hull $B$ of the sequence is compact. Since $E_B$ is Banach and $E_n$ is locally complete, we have that $E_B \cap E_n$ is closed in $E_B$, for every $n$, since every sequence converging in $E_B$ is a Mackey-Cauchy sequence in $E$. Since

$$E_B = \bigcup \{(E_B \cap E_n) : n \in \mathbb{N}\}$$

we have that, by using Baire's Category Theorem, there is a positive integer $n_0 \in \mathbb{N}$ such that $E_B \cap E_{n_0} \supset E_B$, and, therefore, $B \subset E_{n_0}$. That implies $\bar{f} \in C_0(E_{n_0})$.

(3.3.2) follows from the barrelledness, (3.3.1) and the following result of Valdivia ([8]): If a barrelled space is the union of an increasing sequence of subspaces, then such space is the inductive limit of that sequence. Q.e.d.

3.4. Remark. The condition "*locally complete*" for the space $E$ in 3.3. Theorem can not be eliminated, since if $E$ is the non-complete LB-space constructed by Köthe in ([3], p. 434), and if $E_n$, $n = 1, 2, \ldots$ is the increasing sequence of subspaces of $E$, such that there is a topology $\tau_n$, finer than the induced topology by the one of $E$, with $E_n[\tau_n]$ topologically isomorphic to $C_0$, $n = 1, 2, \ldots$, then

$$C_0(E) \neq \bigcup \{C_0(E_n) : n \in \mathbb{N}\}$$

since there is a null sequence $f_n$ in $E$, such that $f_n \notin E_n$.

3.5. Note. Conditions under which a locally convex space $E$, which is the union of an increasing sequence of subspaces, is locally complete, are given in Corollary 2.4 of [4], concerning the hypothesis of 3.3. Theorem.

3.6. Remark. If $E$ is a locally convex space, then there is a topological isomorphism between $C_0(E)$ and $C_0(E) \times E$. This isomorphism can be constructed from the topological isomorphism that exists between $C_0$ and $C_0 \times \mathbb{K}$. In particular, $E$ is a quotient space of $C_0(E)$, as it was obtained in the proof of 2.6. Theorem.

Acknowledgements. I am grateful to the referee for suggesting a number of improvements, specially in the third section of the exposition, and for the reference [9].

## References

[1] H. G. Garnir, Dual d'un espace d'operateurs linéaires. Math. Ann. **163**, 4–8 (1966).
[2] F. Lopez Fernandez-Asenjo, A. Marquina, and A. Perez Gomez, Topologías inductivas sobre $C_0(E)$. Actas de IV Jornadas Matematicas Hispano-Lusitanas. JACA (1977).
[3] G. Koethe, Topological Vector Spaces I. Berlin 1969.
[4] A. Marquina, Sobre límites inductivos estrictos numerables. Collectanea Math. **26**, 3–9 (1975).
[5] A. Pietsch, Nuclear Locally Convex Spaces. Berlin 1972.
[6] J. M. Sanz Serna, Espacios de Sucesiones Vectoriales Acotadas. Tesis Doctoral. Valladolid. (1977) (to appear in C.S.I.C.).
[7] J. M. Sanz Serna, Convergencia débil y fuerte en el dual de $C_0(E)$. Actas de la Reunión de Mátematicos de Expresión Latina. Palma de Mallorca (Spain) (1977).
[8] M. Valdivia, Absolutely convex sets in barrelled spaces. Ann. Inst. Fourier **21**, 3–13 (1971).
[9] H. G. Garnir, M. de Wilde, and J. Schmets, Analyse Fonctionelle I. Basel 1968.

Eingegangen am 10. 5. 1978 *)

Anschrift der Autoren:

A. Marquina
Facultad de Matematicas
Av. Blasco Ibañez, 13
Valencia-10 (Spain)

J. M. Sanz Serna
Facultad de Ciencias
Prado de la Magdalena S/N
Valladolid (Spain)

*) Eine Neufassung ging am 27. 10. 1978 ein.