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A GALERKIN METHOD FOR A NONLINEAR INTEGRO-DIFFERENTIAL WAVE SYSTEM

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A Galerkin method is introduced for the numerical solution of a system equivalent to the equation $w_{tt} = (1 + \int_0^\pi w_x^2 dx) w_{xx}$ which models the vibrations of a string when the increase in tension due to extension is taken into account. We prove the convergence of the semidiscrete approximations and present some numerical tests.

1. Introduction

It is well known that the familiar wave equation $w_{tt} = w_{xx}$ constitutes only a first approximation in the study of the transversal vibrations of a string [1]. The dimensionless nonlinear integro-differential equation

$$w_{tt} = \left(1 + \int_0^\pi w_x^2 dx\right) w_{xx}, \quad 0 < x < \pi, \ t > 0 \tag{1.1}$$

provides a better model in that it takes into account the increase in tension resulting from the extension of the string [2]. Similar nonlinear integro-differential corrections have also been suggested in the study of beams and plates [3, 4].

In this paper we present a Galerkin method for the solution of (1.1) rewritten as the first-order system

$$u_t = \left(1 + \int_0^\pi v^2 dx\right) v_x, \qquad v_t = u_x, \tag{1.2a}$$

which arises from the substitutions $u = w_t$ and $v = w_x$.

Along with (1.2a) we have the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad t > 0 \tag{1.2b}$$

(corresponding to the string being fixed at both ends), and the initial conditions

$$u(x,0), \qquad v(x,0), \quad 0<x<\pi. \tag{1.2c}$$

A semidiscrete (continuous in time) Galerkin method for the numerical solution of (1.2) is introduced in Section 2 and its convergence is proved in Section 3. In Section 4 we discuss some methods for the integration in time of the semidiscrete system. Finally, numerical experiments are reported in Section 5.

The modifications required in order to deal with boundary conditions more general than (1.2b) or with a forcing term in (1.1), (1.2a) are straightforward and there is no need to present them here.

It should be pointed out that for (1.2a) subject to (1.2b) the law of conservation of energy holds under the form (see [2])

$$\int_0^\pi u^2 dx + \int_0^\pi v^2 dx + \tfrac{1}{2}\left(\int_0^\pi v^2 dx\right)^2 = \text{constant}.$$

2. Galerkin method

The interval $[0,\pi]$ is divided into N elements of equal length $h=\pi/N$, by means of a mesh $x_i = ih$, $i=0,1,\ldots,N$. We consider the space S^h of real, continuous, piecewise linear functions in $[0,\pi]$ and the subspace S_0^h consisting of the functions in S^h that vanish at $x=0$ and $x=\pi$.

The Galerkin approximation of u and v is a mapping $t\to(U(t), V(t))$; $U(t)\in S_0^h$, $V(t)\in S^h$ defined by

$$(U_t,\phi) = \left(\left(1+\int_0^\pi V^2 dx\right)V_x, \phi\right), \quad \phi\in S_0^h, \tag{2.1}$$

$$(V_t,\psi) = (U_x,\psi), \qquad \psi\in S^h, \tag{2.2}$$

$$U(0)-u(\cdot\,,0), \qquad V(0)-v(\cdot\,,0) \text{ 'small'}. \tag{2.3}$$

We denote by $(\cdot\,,\cdot)$, $\|\cdot\|$ the inner product and norm in $L^2(0,\pi)$, respectively.

Upon noticing that (2.1), (2.2) is essentially a system of nonlinear differential equations in $2N$ unknowns ($2N=\dim(S_0^h\times S^h)$) we can prove the following result.

THEOREM 1.1. For any value of $h=\pi/N$, N a positive integer and any choice of initial approximations $U(0)\in S_0^h$, $V(0)\in S^h$, there exists a unique Galerkin solution satisfying (2.1), (2.2) *for $0\le t<\infty$. Furthermore, the following a priori L^2-estimate holds*:

$$\|U(t)\|^2 + \|V(t)\|^2 + \tfrac{1}{2}\|V(t)\|^4 = \textit{constant}. \tag{2.4}$$

PROOF. It is sufficient to prove (2.4). From (2.1), (2.2),

$$\tfrac{1}{2}\frac{d}{dt}(\|U(t)\|^2+\|V(t)\|^2+\tfrac{1}{2}\|V(t)\|^4)=(U_t, U)+(V_t, V)+\|V(t)\|^2(V_t, V)$$
$$=(1+\|V(t)\|^2)[(U_x, V)+(V_x, U)],$$

and integration by parts shows that this last term is zero.

Notice that (2.4) establishes the conservation of mechanical energy.

When the usual basis functions $\phi_i \in S^h$, $i=0,1,\ldots,N$, $\phi_i(x_i)=1$, $\phi_i(x_j)=0$, $i\neq j$ are introduced, the approximations $U(t)$, $V(t)$ can be written in the form $U(t)=\sum_{i=1}^{N-1} U_i(t)\phi_i$, $V(t)=\sum_{i=0}^{N} V_i(t)\phi_i$, where $U_i(t)$, $V_i(t)$ are the nodal values of $U(t)$, $V(t)$, respectively. Conditions (2.1), (2.2) are readily seen to be equivalent to the following system of ordinary differential equations for the nodal values:

$$M\frac{d}{dt}\boldsymbol{U}=(1+h\boldsymbol{V}^tN\boldsymbol{V})Q\boldsymbol{V},$$
$$N\frac{d}{dt}\boldsymbol{V}=-Q^t\boldsymbol{U} \tag{2.5}$$

where $\boldsymbol{U}=[U_1, U_2, \ldots, U_{N-1}]^t$, $\boldsymbol{V}=[V_0, V_1, \ldots, V_N]^t$. The matrices M, N, Q have dimensions $(N-1)\times(N-1)$, $(N+1)\times(N+1)$, $(N-1)\times(N+1)$, respectively, and are given by

$$M=\tfrac{1}{6}\begin{bmatrix}4&1&&&\\1&4&1&&\\&\ddots&\ddots&\ddots&\\&&1&4&1\\&&&1&4\end{bmatrix}=h^{-1}((\phi_i,\phi_j))_{1\le i,j\le N-1},$$

$$N=\tfrac{1}{6}\begin{bmatrix}2&1&&&\\1&4&1&&\\&\ddots&\ddots&\ddots&\\&&1&4&1\\&&&1&2\end{bmatrix}=h^{-1}((\phi_i,\phi_j))_{0\le i,j\le N},$$

$$Q=\frac{1}{2h}\begin{bmatrix}-1&0&1&&\\&-1&0&1&\\&&\ddots&\ddots&\ddots\\&&&-1&0&1\end{bmatrix}=h^{-1}((\phi_i',\phi_j))_{1\le i\le N-1,\,0\le j\le N}.$$

3. Convergence

It is well known that for first-order hyperbolic systems, finite element analysis leads in general to suboptimal convergence estimates in L^2 [5]. It is therefore advisable to analyze the

Galerkin method presented in Section 2 in its finite difference formulation (2.5) (see [6]). This will be seen to yield an optimal error estimate.

Let $\boldsymbol{Y}$, $\boldsymbol{Z}$ be $(N-1)$ vectors whose entries Y_i, Z_i, $i = 1, 2, \ldots, N-1$, are values of certain functions at the nodes x_i. We set

$$(\boldsymbol{Y}, \boldsymbol{Z})_h = h \sum_{i=1}^{N-1} Y_i Z_i, \qquad \|\boldsymbol{Y}\|_h^2 = (\boldsymbol{Y}, \boldsymbol{Y})_h.$$

Analogously, if $\boldsymbol{Y}$, $\boldsymbol{Z}$ are $(N+1)$ vectors whose entries Y_i, Z_i, $i = 0, 1, \ldots, N$, are function values at the nodes x_i, we set

$$(\boldsymbol{Y}, \boldsymbol{Z})_h = h \sum_{i=0}^{N} Y_i Z_i, \qquad \|\boldsymbol{Y}\|_h^2 = (\boldsymbol{Y}, \boldsymbol{Y})_h.$$

The bounds

$$\tfrac{1}{3}\|\boldsymbol{Y}\|_h^2 \leq (\boldsymbol{Y}, M\boldsymbol{Y})_h \leq \|\boldsymbol{Y}\|_h^2, \quad \boldsymbol{Y} \in \mathbb{R}^{N-1}, \tag{3.1a}$$

$$\tfrac{1}{6}\|\boldsymbol{Y}\|_h^2 \leq (\boldsymbol{Y}, N\boldsymbol{Y})_h \leq \|\boldsymbol{Y}\|_h^2, \quad \boldsymbol{Y} \in \mathbb{R}^{N+1}, \tag{3.1b}$$

$$\|Q^t\boldsymbol{Y}\|_h \leq h^{-1}\|\boldsymbol{Y}\|_h, \qquad \boldsymbol{Y} \in \mathbb{R}^{N-1}, \tag{3.1c}$$

follow easily from Gerschgorin's theorem applied to M, N, QQ^t, respectively.

We assume that (1.2) has a unique solution (u, v), with $u, v \in C^3([0, \pi] \times [0, T])$ and denote by $\boldsymbol{u} = \boldsymbol{u}(t)$, $\boldsymbol{v} = \boldsymbol{v}(t)$ the corresponding vectors of nodal values $[u(x_1, t), \ldots, u(x_{N-1}, t)]^t$, $[v(x_0, t), \ldots, v(x_N, t)]^t$. The vectors of truncation errors $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t)$ are defined by the equations

$$M\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = (1 + h\boldsymbol{v}^tN\boldsymbol{v})Q\boldsymbol{v} + \boldsymbol{\rho}, \tag{3.2a}$$

$$N\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = -Q^t\boldsymbol{u} + \boldsymbol{\sigma}. \tag{3.2b}$$

Note that for any $(N+1)$ vector $\boldsymbol{Z}$ the product $h\boldsymbol{Z}^tN\boldsymbol{Z} = (\boldsymbol{Z}, N\boldsymbol{Z})_h$ equals the square of the $L^2(0, \pi)$-norm of the element in S^h with nodal values Z_i. Thus in (3.2a) the product $h\boldsymbol{v}^tN\boldsymbol{v}$ is, for each value of t, the square of the norm of the interpolant $\hat{v}$ of v. Accordingly, $h\boldsymbol{v}^tN\boldsymbol{v} = \|v(\cdot, t)\|^2 + O(h^2)$. Now a Taylor expansion reveals that $\|\boldsymbol{\rho}(t)\|_h + \|\boldsymbol{\sigma}(t)\|_h \leq Ch^2$, $0 \leq t \leq T$, where C is a constant depending only on bounds of the first, second and third derivatives of u and v in $[0, \pi] \times [0, T]$.

THEOREM 3.1 Assume that u, v satisfy the hypotheses above and that $\|\boldsymbol{u}(0) - \boldsymbol{U}(0)\|_h + \|\boldsymbol{v}(0) - \boldsymbol{V}(0)\|_h \leq C_1h^2$. *Then there exist positive constants* C_2, C_3 *depending on* C_1, T *and on bounds of the derivatives of u and v in* $[0, \pi] \times [0, T]$ *such that for* $h \leq C_2$ *and* $0 \leq t \leq T$,

$$\|\boldsymbol{u}(t) - \boldsymbol{U}(t)\|_h + \|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_h \leq C_3h^2. \tag{3.3}$$

PROOF. It is sufficient to carry out the proof under the a priori assumption

$$\|\boldsymbol{u}(t)-\boldsymbol{U}(t)\|_h \leq h\,, \tag{3.4}$$

as this can be removed by means of a standard argument. We shall denote by C, a generic constant depending only on C_1, T and bounds for the derivatives of u, v.

The energy method based on the functional

$$E(t)=(\boldsymbol{u}-\boldsymbol{U}, M(\boldsymbol{u}-\boldsymbol{U}))_h+[1+(\boldsymbol{V}, N\boldsymbol{V})_h](\boldsymbol{v}-\boldsymbol{V}, N(\boldsymbol{v}-\boldsymbol{V}))_h$$

will be employed. Note that from (3.1)

$$(\boldsymbol{V}(0), N\boldsymbol{V}(0))_h \leq \|\boldsymbol{V}(0)\|_h^2 \leq 2\|\boldsymbol{v}(0)\|_h^2+2\|\boldsymbol{v}(0)-\boldsymbol{V}(0)\|_h^2\,,$$

and the last expression can be bounded independently of h. On considering the identity $(\boldsymbol{V}, N\boldsymbol{V})_h=\|\boldsymbol{V}\|^2$ and the a priori estimate (2.4), we conclude that $(\boldsymbol{V}, N\boldsymbol{V})_h$ can be bounded independently of h for $0\leq t\leq T$. Therefore (3.1) implies the existence of positive constants A, B independent of h such that

$$AE^*(t)\leq E(t)\leq BE^*(t) \tag{3.5}$$

where

$$E^*(t)=\|\boldsymbol{u}(t)-\boldsymbol{U}(t)\|_h^2+\|\boldsymbol{v}(t)-\boldsymbol{V}(t)\|_h^2\,.$$

Differentiation of $E(t)$, the use of (2.4), (3.2) to eliminate $\boldsymbol{u}_t$, $\boldsymbol{v}_t$, $\boldsymbol{U}_t$, $\boldsymbol{V}_t$ and some rearrangements yield

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(t)=&[(\boldsymbol{v}, N\boldsymbol{v})_h-(\boldsymbol{V}, N\boldsymbol{V})_h](\boldsymbol{u}-\boldsymbol{U}, Q\boldsymbol{v})_h+(\boldsymbol{u}-\boldsymbol{U}, \boldsymbol{\rho})_h\\&+[1+(\boldsymbol{V}, N\boldsymbol{V})_h](\boldsymbol{v}-\boldsymbol{V}, \boldsymbol{\sigma})_h-(\boldsymbol{V}, Q^t\boldsymbol{U})_h(\boldsymbol{v}-\boldsymbol{V}, N(\boldsymbol{v}-\boldsymbol{V}))_h\,.\end{aligned} \tag{3.6}$$

We now estimate the right-hand side of (3.6) as follows: First

$$(\boldsymbol{u}-\boldsymbol{U}, \boldsymbol{\rho})_h \leq \tfrac{1}{2}(\|\boldsymbol{u}-\boldsymbol{U}\|_h^2+\|\boldsymbol{\rho}\|_h^2) \tag{3.7}$$

and

$$[1+(\boldsymbol{V}, N\boldsymbol{V})_h](\boldsymbol{v}-\boldsymbol{V}, \boldsymbol{\sigma})_h \leq [\tfrac{1}{2}(1+D)](\|\boldsymbol{v}-\boldsymbol{V}\|_h^2+\|\boldsymbol{\sigma}\|_h^2) \tag{3.8}$$

where D is an a priori bound for $(\boldsymbol{V}, N\boldsymbol{V})_h$. Next we write

$$\begin{aligned}|(\boldsymbol{V}, Q^t\boldsymbol{U})_h| &\leq \|\boldsymbol{V}\|_h\|Q^t\boldsymbol{U}\|_h \leq C\|Q^t\boldsymbol{U}\|_h\\&\leq C(\|Q^t(\boldsymbol{u}-\boldsymbol{U})\|_h+\|Q^t\boldsymbol{u}\|_h)\,,\end{aligned}$$

and observe that $\|Q^t\boldsymbol{u}\|_h$ can be bounded independently of h. Thus, on considering (3.1c) and (3.4) we conclude that

$$|(\boldsymbol{V}, Q^t\boldsymbol{U})_h| \leq C,$$

whence,

$$|(\boldsymbol{V}, Q^t\boldsymbol{U})_h(\boldsymbol{v}-\boldsymbol{V}, N(\boldsymbol{v}-\boldsymbol{V}))_h| \leq C\|\boldsymbol{v}-\boldsymbol{V}\|_h^2. \tag{3.9}$$

Finally,

$$(\boldsymbol{u}-\boldsymbol{U}, Q\boldsymbol{v})_h \leq C\|\boldsymbol{u}-\boldsymbol{U}\|_h$$

and

$$(\boldsymbol{v}, N\boldsymbol{v})_h - (\boldsymbol{V}, N\boldsymbol{V})_h = (\boldsymbol{v}+\boldsymbol{V}, N(\boldsymbol{v}-\boldsymbol{V}))_h \leq C\|\boldsymbol{v}-\boldsymbol{V}\|_h. \tag{3.10}$$

Substitution of (3.7)–(3.10) in (3.6) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq C(E^*(t) + \|\boldsymbol{\rho}\|_h^2 + \|\boldsymbol{\sigma}\|_h^2) \leq C(E(t) + h^4),$$

and now Gronwall's lemma can be used to complete the proof.

It should be pointed out that (3.3) clearly implies the optimal estimate in the $L^2[0, \pi]$-norm:

$$\|u(\cdot, t) - U(t)\| + \|v(\cdot, t) - V(t)\| = \mathrm{O}(h^2), \quad 0 \leq t \leq T.$$

4. Integration in time

Care should be exercised in the time integration of the system (2.5) as this is not a *banded* system as a result of the presence of the 'nonlocal' term $\boldsymbol{V}^t\boldsymbol{N}\boldsymbol{V}$.

In our experiments, a predictor-corrector algorithm was used in order to avoid the need for full matrices. We denote by k the step size in time and by $\boldsymbol{U}^n$, $\boldsymbol{V}^n$ the approximations of $\boldsymbol{U}(nk)$, $\boldsymbol{V}(nk)$, respectively. A step of the algorithm consists of a predictor stage

$$N(\boldsymbol{V}_{[0]} - \boldsymbol{V}^n) = -kQ^t\boldsymbol{U}^n \tag{4.1}$$

and the corrector stages

$$\begin{aligned} M(\boldsymbol{U}_{[s]} - \boldsymbol{U}^n) &= \tfrac{1}{2}k[(1 + F(\boldsymbol{V}_{[s-1]}))Q\boldsymbol{V}_{[s]} + (1 + F(\boldsymbol{V}^n))Q\boldsymbol{V}^n], \\ N(\boldsymbol{V}_{[s]} - \boldsymbol{V}^n) &= -\tfrac{1}{2}k(Q^t\boldsymbol{U}_{[s]} + Q^t\boldsymbol{U}^n). \end{aligned} \tag{4.2}$$

Here, $s = 1, 2, \ldots$, $F(\boldsymbol{Z}) = h\boldsymbol{Z}^t\boldsymbol{N}\boldsymbol{Z}$. Formulas (4.2) are applied until a value of s is found for which the iterates s, $s-1$ are identical within the accuracy of the machine. Then $\boldsymbol{U}^{n+1}$, $\boldsymbol{V}^{n+1}$ are taken to be $\boldsymbol{U}_{[s]}$, $\boldsymbol{V}_{[s]}$ respectively.

The matrix N can be factorized once and for all and therefore an application of the predictor demands a backward and a forward substitution on an $(N+1)$ vector. Each corrector stage requires the solution of a linear system in $2N$ unknowns, whose matrix changes with n and s. This matrix is block-tridiagonal with 2×2 blocks, provided that the unknowns are taken in the order V_0, U_1, V_1, U_2, $V_2, \ldots, V_N$.

Clearly the accuracy and stability properties of the algorithm (4.1), (4.2) are identical, except

for round-off [7, p. 86], with those of the *implicit* Crank–Nicolson method

$$\begin{aligned} M(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n) &= \tfrac{1}{2}k[(1+F(\boldsymbol{V}^{n+1}))Q\boldsymbol{V}^{n+1}+(1+F(\boldsymbol{V}^n))Q\boldsymbol{V}^n]\,, \\ N(\boldsymbol{V}^{n+1}-\boldsymbol{V}^n) &= -\tfrac{1}{2}k(Q^t\boldsymbol{U}^{n+1}+Q^t\boldsymbol{U}^n)\,, \end{aligned} \tag{4.3}$$

the predictor-corrector procedure being simply a technique for the solution of the implicit equations for $\boldsymbol{U}^{n+1}$, $\boldsymbol{V}^{n+1}$. The use of Newton's method in (4.3) is not recommended as the relevant Jacobian matrix is not sparse due to the term $F(\boldsymbol{V}^{n+1})$.

Some computational effort can be saved by allowing only one application of the corrector per time step. In that case the resulting algorithm is not identical to (4.3) and nonlinear instabilities may threaten. See the experiments in [8], based on a similar procedure, and the discussion in [9, 10].

An alternative to (4.3) is given by the modified Crank–Nicolson method [10, 11, 12]:

$$M(\boldsymbol{U}^{n+1}-\boldsymbol{U}^n) = \tfrac{1}{4}k(2+F(\boldsymbol{V}^{n+1})+F(\boldsymbol{V}^n))Q(\boldsymbol{V}^{n+1}+\boldsymbol{V}^n)\,, \tag{4.4a}$$

$$N(\boldsymbol{V}^{n+1}-\boldsymbol{V}^n) = -\tfrac{1}{2}k(Q^t\boldsymbol{U}^{n+1}+Q^t\boldsymbol{U}^n)\,, \tag{4.4b}$$

which possesses the conservation property (cf. (2.4))

$$(\boldsymbol{U}^n, M\boldsymbol{U}^n)_h + (\boldsymbol{V}^n, N\boldsymbol{V}^n)_h + \tfrac{1}{2}(\boldsymbol{V}^n, N\boldsymbol{V}^n)_h^2 = \text{constant}\,.$$

This is derived by taking the scalar products of (4.4a) with $(\boldsymbol{U}^{n+1}+\boldsymbol{U}^n)$ and of (4.4b) with $(\boldsymbol{V}^{n+1}+\boldsymbol{V}^n)$ and adding the results. The importance of conserving discrete analogues of continuous invariants of motion is now widely appreciated (see the discussion in [9] and the literature cited there).

The local accuracy of both (4.3) and (4.4) is $O(k^2+h^2)$. Convergence proofs of these fully discrete methods will not be presented in this paper.

5. Numerical experiments

We applied the method outlined in Section 2 to the problem given by the system (1.2) with initial conditions

$$u(x,0)=0\,, \qquad v(x,0)=0.25\cos x\,. \tag{5.1}$$

The exact solution is given by

$$u(x,t)=\dot{\alpha}(t)\sin x\,, \qquad v(x,t)=\alpha(t)\cos x \tag{5.2}$$

where dot denotes differentiation. The function $\alpha(t)$ satisfies the Duffing equation

$$\ddot{\alpha}+\alpha+\tfrac{1}{2}\pi\alpha^3=0\,, \tag{5.3}$$

Table 1
Error $\times 10^6$ at $x = 0.5$, $t = 5$

	$k = 0.001$	$k = 0.05$
$h = \pi/10$	235.1	370.1
$h = \pi/40$	13.0	149.3
$h = \pi/160$	0.8	136.4

with initial conditions $\alpha(0) = 0.25$, $\dot{\alpha}(0) = 0$. The predictor-corrector (4.1), (4.2) was employed to march in time. Two different values $k = 0.05$, 0.001 were used for the time step. When $k = 0.001$, the error due to the time integration is negligible and accordingly the computed solution is virtually identical to the solution of the semidiscrete system (2.5). The value $k = 0.05$ was chosen to assess the effect of the integration in time. For this latter value of k the convergence of the corrector needed two or three iterations per time step.

Table 1 displays the error (numerical − theoretical) associated with the approximation $u(0.5, 5) \sim 0.2301349$. This 'exact' value was computed by solving (5.3) with a standard ODE package.

The column $k = 0.001$ exhibits an $O(h^2)$ behavior. Each entry in the column $k = 0.05$ approximately exceeds the corresponding entry on the left by the fixed quantity 135. This is explained by the fact that the error in the fully discrete method (4.1), (4.2) behaves like $O(k^2 + h^2)$, 135×10^{-6} being the $O(k^2)$ term corresponding to $k = 0.05$.

Other nodal errors in the u and v components follow the pattern presented in the table for the error in u at $x = 0.5$, $t = 5$.

Experiments involving up to 20000 time steps ($h = \frac{1}{40}\pi$, $k = 0.0125$) and 'rough' initial data were also performed without stability problems.

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