Computing 44, 187–196 (1990)

Computing

The Spectral Accuracy of a Fully-Discrete Scheme for a Nonlinear Third Order Equation

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Dedicated to Professor Hans J. Stetter on the occasion of his 60th birthday

Received September 21, 1989

Abstract — Zusammenfassung

The Spectral Accuracy of a Fully-Discrete Scheme for a Nonlinear Third Order Equation. A time-discrete pseudospectral algorithm is suggested for the numerical solution of a nonlinear third order equation arising in fluidization. The nonlinear stability and convergence of the new scheme are analyzed. Numerical comparisons with available finite-difference methods are also reported which clearly indicate the superiority of the new scheme.

AMS Subject Classification: 65M10

Key words: Spectral method, stability, stability threshold

Die spektrale Genauigkeit eines voll diskreten Schemas für eine nichtlineare Gleichung dritter Ordnung. Zur numerischen Lösung einer nichtlinearen Differentialgleichung dritter Ordnung, herrührend aus einem Strömungsproblem bei Gasteilchen, wird ein zeitdiskreter Pseudo-spektral-Algorithmus vorgeschlagen. Stabilität und Konvergenz des neuen Differenzenverfahrens werden analysiert. Numerische Vergleiche mit bestehenden Differenzenschemata sprechen klar zugunsten des neuen Verfahrens.

1. Introduction

We are concerned with the nonlinear, periodic initial-value problem

$$u_t + u_{xxx} + \beta(u^2)_x + (\gamma/2)(u^2)_{xx} + \varepsilon u_{xx} - \delta u_{tx} = 0, \quad -\infty < x < \infty, \quad 0 < t \le T < \infty, \tag{1.1}$$

$$u(x,t) = u(x+2\pi, t), \quad -\infty < x < \infty, \quad 0 \le t \le T, \tag{1.2}$$

$$u(x,0) = q(x), \tag{1.3}$$

where β, γ, ε, δ are given real constants with $\varepsilon, \delta > 0$, the unknown u is real-valued and the given function q is 2π-periodic. The problem (1.1)–(1.3) arises in the theory of flow in a gas-particle fluidized bed [6] with u representing the value of a spatially periodic small perturbation of the concentration of particles. Christie & Ganser [3] have numerically studied (1.1)–(1.3) by means of finite-difference and modified-Galerkin methods. These authors discovered that the numerical integration of (1.1)–(1.3) is a difficult task, due to the delicate balance between the various terms in (1.1); a balance which is likely to be destroyed by the discretization procedure, resulting in an unstable scheme. In fact many 'reasonable' time-implicit schemes perform in an unexpectedly unstable manner, while the application of other implicit schemes leads to stable computations only if the time step is *large* enough relative to the mesh-size in space. In [1] Christie and the present authors have analyzed the numerical difficulties encountered in [3] and suggested a well-behaved finite-difference scheme. Since the problem (1.1)–(1.3) is periodic, it is only natural to ask whether Fourier spectral and pseudospectral techniques can be successfully applied. In this paper we suggest a fully discrete pseudospectral scheme for (1.1)–(1.3) and prove that it produces spectrally small spatial errors.

Our analysis uses the general framework of [8], [9], whose key feature is a definition of stability for nonlinear problems that employs the notion of h-dependent stability thresholds. The idea of stability threshold goes back to Stetter. In his 1966 paper [10], Stetter, after referring as "stable" to situations where "the global effect ε of a local perturbation δ remains bounded in terms of δ", noticed that "we cannot expect a nonlinear discretization to remain well behaved for large perturbations" and thus was led to the concept of stability threshold. A stability threshold for a nonlinear discretization is an upper bound such that the discretization is well behaved for perturbations whose size is below the threshold. Furthermore, Stetter pointed out [10] that "the stability thresholds *may* decrease with a power of h", the discretization parameter. However, for the discretizations that arise in the numerical study of ordinary differential equations (ODEs), the thresholds *do not* actually decrease with h so that they may be chosen to be independent of h. For this reason, the general framework considered in Stetter's ODE book [11] does not allow the h-dependence of the thresholds. Similarly, H. B. Keller's theory of ODE discretizations [7], which also uses thresholds, does not cater for the dependence of the thresholds on h. Nevertheless, h-dependent stability thresholds are essential in the study of numerical partial differential equations (PDEs) [10], [5] and therefore were included in the theory of discretizations developed in [8], [9] by J. C. López-Marcos and one of the present authors. In a sense, the material in [8], [9] can be seen as a generalization of the frameworks of [7], [11] so as to also cover the PDE case (see [8]).

It is also appropriate to note that, as shown in [5], the consideration of h-dependent stability thresholds avoids the need for a priori estimates in convergence proofs of nonlinear algorithms. This simplification is possible by using the main theorem in [8], [9] which is based on a deep result due to Stetter [11].

An outline of the present paper is as follows. The new scheme is presented in section 2 and analyzed in section 3. The final section is devoted to some numerical illustrations.

The article [4] also presents a convergence proof of a spectral algorithm within the framework of [8], [9].

2. Numerical Method

We first need some notation. If J is a positive integer, we set $h = 2\pi/(2J)$ and consider the grid-points $x_j = jh, j = 0, \pm 1, \pm 2, \ldots$ We denote by $\mathbb{Z}_h$ the space of 2π-periodic real functions defined on the grid. Thus, each element $\mathbf{V} \in \mathbb{Z}_h$ is a real sequence $\{V_j\}_{j=0,\pm1,\ldots}$ such that $V_j = V_{j+2J}$, $j = 0, \pm 1, \ldots$. The notation $[\mathbf{V}]_p^\wedge$ refers to the p-th discrete Fourier coefficient of the sequence $\mathbf{V}$, i.e.

$$[\mathbf{V}]_p^\wedge = (1/2\pi) \sum_{0 \le j \le 2J}'' hV_j \exp(-ipjh), \qquad -J \le p \le J,$$

where the double prime in the summation means that the first and last termed are halved. To recover $\mathbf{V}$ from its Fourier coefficients it is enough to evaluate at the grid points the trigonometric interpolant $V^*(x)$ of $\mathbf{V}$ given by

$$V^*(x) = \sum_{-J \le p \le J}'' [\mathbf{V}]_p^\wedge \exp(ipx), \qquad -\infty < x < \infty.$$

On differentiating this identity and evaluating the result at the grid points we obtain the following definition of the spectral difference operator D, mapping $\mathbb{Z}_h$ into itself

$$(2.1) \qquad (D\mathbf{V})_j = \sum_{-J \le p \le J}'' [\mathbf{V}]_p^\wedge (ip) \exp(ipjh), \qquad \mathbf{V} \in \mathbb{Z}_h, \qquad j = 0, 1, \ldots$$

The relation (2.1) is of course equivalent to the following simple formula for the Fourier coefficients

$$(2.2) \qquad [D\mathbf{V}]_p^\wedge = (ip)[\mathbf{V}]_p^\wedge, \qquad -J \le p \le J.$$

The powers $D^2, D^3, \ldots$ of D are, by definition, the composite operators DD, DDD, ... The product $\mathbf{VW}$ of elements of $\mathbb{Z}_h$ is to be interpreted componentwise.

With this notation, we discretize in space the problem (1.1)–(1.3) and look for a mapping $\mathbf{U}$: $[0, T] \to \mathbb{Z}_h$ such that $\mathbf{U}(0)$ is an approximation to the (grid restriction of the) initial datum q and, for $0 \le t \le T$,

(2.3)
$$(d/dt)\mathbf{U}(t) + D^3\mathbf{U}(t) + \beta D\mathbf{U}^2(t) + (\gamma/2)D^2\mathbf{U}^2(t) + \varepsilon D^2\mathbf{U}(t) - \delta D(d/dt)\mathbf{U}(t) = \mathbf{0}.$$

Here, U(t) approximates the grid restriction of the solution $u(\cdot, t)$. The time-continuous scheme (2.3) has been considered in [2].

For the discretization in time of (2.3), we denote by k the time-step $0 < k < T$ and consider the time-levels $t_n = nk$, $n = 0, 1, \ldots, N = [T/k]$. The approximations $\mathbf{U}^n$ to $\mathbf{u}^n$, grid restriction of $u(\cdot, t_n)$, are then computed by using the three-level scheme

$$(2.4) \qquad \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{k} + D^3 \frac{\mathbf{U}^{n+1} + \mathbf{U}^n}{2} + \beta D\left(\frac{3}{2}(\mathbf{U}^n)^2 - \frac{1}{2}(\mathbf{U}^{n-1})^2\right)$$
$$+ \frac{\gamma}{2} D^2 \left(\frac{3}{2}(\mathbf{U}^n)^2 - \frac{1}{2}(\mathbf{U}^{n-1})^2\right) + \varepsilon D^2 \left(\frac{3}{3}\mathbf{U}^n - \frac{1}{2}\mathbf{U}^{n-1}\right)$$
$$- \delta D \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{k} = \mathbf{0}, \qquad 1 \le n \le N - 1,$$

which can be seen as a blend between the standard trapezoidal and two-step Adams-Bashforth methods. The nonlinear terms and the term with ε are treated explicitly, while the stiffest term arising from u_{xxx} is dealt with in an implicit manner.

For implementation purposes it is best to transform (2.4) to obtain, on taking into account (2.2),

$$\begin{aligned}(2.5)\qquad &\frac{Y_p^{n+1}-Y_p^n}{k}+(ip)^3\frac{Y_p^{n+1}+Y_p^n}{2}+\beta(ip)\left(\frac{3}{2}Z_p^n-\frac{1}{2}Z_p^{n-1}\right)\\ &+\frac{\gamma}{2}(ip)^2\left(\frac{3}{2}Z_p^n-\frac{1}{2}Z_p^{n-1}\right)+\varepsilon(ip)^2\left(\frac{3}{2}Y_p^n-\frac{1}{2}Y_p^{n-1}\right)\\ &-\delta(ip)\frac{Y_p^{n+1}-Y_p^n}{k}=0,\qquad -J\le p\le J,\qquad 1\le n\le N-1,\end{aligned}$$

where Y_p^n, Z_p^n respectively denote the p-th Fourier coefficients $[\mathbf{U}^n]_p^\wedge\, y[(\mathbf{U}^n)^2]_p^\wedge$. Note that in (2.5) there is no coupling between different wave-numbers p. Therefore there are no systems of linear equations to be solved. The step $n\to n+1$ in (2.5), once Y_p^n, Y_p^{n-1} and Z_p^{n-1}, $-J\le p\le J$, are known, requires an inverse Fourier transform (to obtain $\mathbf{U}^n$ from Y_p^n, $-J\le p\le J$) and a Fourier transform (to obtain Z_p^n, $-J\le p\le J$, from $(\mathbf{U}^n)^2$).

3. Analysis

We first construct the energy norm which will be used later in the stability and convergence analysis of (2.4).

Let D^{-1} represent the operator in $\mathbb{Z}_h$ defined by the relations

$$D(D^{-1}\mathbf{V})=\mathbf{V}-\langle\mathbf{V}\rangle\mathbf{1},\qquad \langle D^{-1}\mathbf{V}\rangle=\langle\mathbf{V}\rangle,$$

where $\langle\cdot\rangle$ denotes mean value (i.e. $\langle\mathbf{V}\rangle=[\mathbf{V}]_0^\wedge$) and $\mathbf{1}$ represents the grid function which takes the value 1 at each grid point. In terms of Fourier coefficients, D^{-1} is defined by the formulas

$$[D^{-1}\mathbf{V}]_p^\wedge=(ip)^{-1}[\mathbf{V}]_p^\wedge,\qquad p=\pm1,\dots,\pm J,\qquad [D^{-1}\mathbf{V}]_0^\wedge=[\mathbf{V}]_0^\wedge.$$

Note that D and D^{-1} commute so that

$$D^{-1}(D\mathbf{V})=\mathbf{V}-\langle\mathbf{V}\rangle\mathbf{1}.$$

The energy norm $\|\cdot\|_E$ in $\mathbb{Z}_h$ is then defined by

$$(3.1)\qquad \|\mathbf{V}\|_{E^2}=\|D^{-1}\mathbf{V}\|^2+\delta^2\|\mathbf{V}\|^2,$$

where $\|\cdot\|$ is the standard discrete L^2-norm

$$\|\mathbf{V}\|^2=\sum_{0\le j\le 2J}'' h(V_j)^2.$$

Note that for each fixed δ the energy norm is equivalent to the discrete L^2-norm uniformly in h.

Nonlinear stability

If $\mathbf{V}^0,\mathbf{V}^1,\dots,\mathbf{V}^N$ and $\mathbf{W}^0,\mathbf{W}^1,\dots,\mathbf{W}^N$ are sequences of elements in $\mathbb{Z}_h$, we define the elements

$$\mathbf{F}^{n+1} = \frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{k} + D^3 \frac{\mathbf{V}^{n+1} + \mathbf{V}^n}{2} + \beta D\left(\frac{3}{2}(\mathbf{V}^n)^2 - \frac{1}{2}(\mathbf{V}^{n-1})^2\right) + \frac{\gamma}{2} D^2\left(\frac{3}{2}(\mathbf{V}^n)^2 - \frac{1}{2}(\mathbf{V}^{n-1})^2\right) + \varepsilon D^2\left(\frac{3}{2}\mathbf{V}^n - \frac{1}{2}\mathbf{V}^{n-1}\right) - \delta D \frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{k}, \quad 1 \le n \le N-1, \tag{3.2}$$

and

$$\mathbf{G}^{n+1} = \frac{\mathbf{W}^{n+1} - \mathbf{W}^n}{k} + D^3 \frac{\mathbf{W}^{n+1} + \mathbf{W}^n}{2} + \beta D\left(\frac{3}{2}(\mathbf{W}^n)^2 - \frac{1}{2}(\mathbf{W}^{n-1})^2\right) + \frac{\gamma}{2} D^2\left(\frac{3}{2}(\mathbf{W}^n)^2 - \frac{1}{2}(\mathbf{W}^{n-1})^2\right) + \varepsilon D^2\left(\frac{3}{2}\mathbf{W}^n - \frac{1}{2}\mathbf{W}^{n-1}\right) - \delta D \frac{\mathbf{W}^{n+1} - \mathbf{W}^n}{k}, \quad 1 \le n \le N-1. \tag{3.3}$$

Thus $\{\mathbf{V}^n\}$, $\{\mathbf{W}^n\}$ can be viewed as solutions of (2.4) after perturbations, with $\{\mathbf{F}^n\}$ and $\{\mathbf{G}^n\}$ being the perturbations. The stability analysis attempts to estimate the distance between $\mathbf{V}^n$ and $\mathbf{W}^n$ in terms of the distance between $\mathbf{F}^n$ and $\mathbf{G}^n$. The following result holds (note the stability thresholds in (3.4)–(3.5)):

Theorem 3.1 (*Stability*). *Assume that the solution u of* (1.1)–(1.3) *is a bounded function for* $0 \le x \le 2\pi$, $0 \le t \le T$, *and set* $M = \max\{|u(x,t)|: 0 \le x \le 2\pi, 0 \le t \le T\}$. *Let* μ *be an arbitrary positive number. There exist positive constants* k_0 *and* C, *depending only on* μ, M, T *and on the parameters* ε, δ, β, γ *of the problem, such that if* $\{\mathbf{V}^n\}$, $\{\mathbf{W}^n\}$, $\{\mathbf{F}^n\}$, $\{\mathbf{G}^n\}$ *are as above and*

$$\max_{0 \le n \le N} \|\mathbf{V}^n - \mathbf{u}^n\|_E \le \mu h^{1/2}, \tag{3.4}$$

$$\max_{0 \le n \le N} \|\mathbf{W}^n - \mathbf{u}^n\|_E \le \mu h^{1/2}, \tag{3.5}$$

then, for $k < k_0$

$$\max_{0 \le n \le N} \|\mathbf{V}^n - \mathbf{W}^n\|_E \le C\Big\{\|\mathbf{V}^0 - \mathbf{W}^0\|_E + \|\mathbf{V}^1 - \mathbf{W}^1\|_E + \sum_{2 \le n \le N} k\|\mathbf{F}^n - \mathbf{G}^n\|_E\Big\} \tag{3.6}$$

Proof. Set $\mathbf{E}^n = \mathbf{V}^n - \mathbf{W}^n$, $0 \le n \le N$, $\mathbf{L}^n = \mathbf{F}^n - \mathbf{G}^n$, $2 \le n \le N$, and subtract (3.3) from (3.2) to arrive at

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{k} + D^3 \frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} + \beta D\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right] + \frac{\gamma}{2} D^2\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right] \tag{3.7}$$

$$+ \varepsilon D^2\left(\frac{3}{2}\mathbf{E}^n - \frac{1}{2}\mathbf{E}^{n-1}\right) - \delta D\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{k} = \mathbf{L}^{n+1}, \quad n = 1, \ldots, N-1.$$

We now apply the operator D^{-1} to (3.7). Before doing so, it should be observed that on taking mean values in (3.7), $\langle(\mathbf{E}^{n+1} - \mathbf{E}^n)/k\rangle = \langle\mathbf{L}^{n+1}\rangle$, while $D^2((\mathbf{E}^{n+1} + \mathbf{E}^n)/2)$, $D[3/2((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - 1/2((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)]$ and $D(3/2\mathbf{E}^n - 1/2\mathbf{E}^{n-1})$ have zero mean. Thus (3.7) implies

(3.8)

$$D^{-1}\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{k} + D^2\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} + \beta\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right]$$
$$- \beta\left\langle\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right\rangle\mathbf{1}$$
$$+ \frac{\gamma}{2}D\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right]$$
$$+ \varepsilon D\left(\frac{3}{2}\mathbf{E}^n - \frac{1}{2}\mathbf{E}^{n-1}\right) - \delta\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{k} + \delta\langle\mathbf{L}^{n+1}\rangle\mathbf{1} = D^{-1}\mathbf{L}^{n+1}, \quad n = 1, \ldots, N-1.$$

On taking the inner product of this formula with $D^{-1}((\mathbf{E}^{n+1} + \mathbf{E}^n)/2) - \delta(\mathbf{E}^{n+1} + \mathbf{E}^n)/2$ and manipulating, we arrive at

(3.9)

$$\left(D^{-1}\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{k} - \delta\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{k}, D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)$$
$$+ \left(D^2\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}, D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)$$
$$= -\beta\left(\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2\right.$$
$$\left. - (\mathbf{W}^{n-1})^2), D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)$$
$$+ \beta\left\langle\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right\rangle$$
$$\times\left(\mathbf{1}, D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right) - \frac{\gamma}{2}\left(D\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2)\right.\right.$$
$$\left.\left. - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right], D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)$$
$$- \varepsilon\left(D\left(\frac{3}{2}\mathbf{E}^n - \frac{1}{2}\mathbf{E}^{n-1}\right), D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)$$
$$+ \left(D^{-1}\mathbf{L}^{n+1} - \delta\langle\mathbf{L}^{n+1}\rangle\mathbf{1}, D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right),$$

$n = 1, \ldots, N-1.$

The definition in (3.1) shows that the first term in the left hand side of (3.9) equals $(1/2k)(\|\mathbf{E}^{n+1}\|_E^2 - \|\mathbf{E}^n\|_E^2)$. The second term equals $(\delta/4)\ \|D(\mathbf{E}^{n+1} + \mathbf{E}^n)\|^2$, since the operator D is skew-symmetric. We shall bound the inner products in the right hand side of (3.9). It is convenient to note first that, from the threshold conditions (3.4)–(3.5),

$$\|\mathbf{V}^n + \mathbf{W}^n\|_\infty \le \|\mathbf{V}^n - \mathbf{u}^n\|_\infty + \|\mathbf{W}^n - \mathbf{u}^n\|_\infty + 2\|\mathbf{u}^n\|_\infty \le 2M + \frac{2\mu}{\delta^2} =: R \tag{3.10}$$

We bound simultaneously the first two terms in the right of (3.9). The Cauchy-Schwartz inequality leads to the bound

$$\left\| \beta\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right] - \beta\left\langle\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) \right.\right.$$
$$\left.\left. - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right\rangle \mathbf{1}\right\| \left\| D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} - \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right\|,$$

Parseval's identity and (3.10) yield the new bounds

$$\left\| \beta\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right]\right\|$$
$$\times \left(\left\| D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} + \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right\|\right)$$
$$\le \left(\frac{3}{4}|\beta|\,\|\mathbf{V}^n + \mathbf{W}^n\|_\infty\|\mathbf{E}^n\| + \frac{1}{4}|\beta|\,\|\mathbf{V}^{n-1} + \mathbf{W}^{n-1}\|_\infty\|\mathbf{E}^{n-1}\|\right)$$
$$\times (\|D^{-1}\mathbf{E}^{n+1}\| + \|D^{-1}\mathbf{E}^n\| + \delta\|\mathbf{E}^{n+1}\| + \delta\|\mathbf{E}^n\|)$$
$$\le \frac{1}{2}|\beta| R\|\mathbf{E}^{n+1}\|_E^2 + \frac{1}{2}|\beta| R\|\mathbf{E}^n\|_E^2 + \frac{3}{2}|\beta| R\|\mathbf{E}^n\|^2 + \frac{1}{2}|\beta| R\|\mathbf{E}^{n-1}\|^2.$$

We next turn to the term with γ. This is split as follows

$$\frac{\gamma}{2}\left(D\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right], D^{-1}\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)$$
$$- \frac{\gamma}{2}\left(D\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right], \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right). \tag{3.11}$$

The first term in (3.11) can be bounded with the technique used for the β terms. For the second inner product in (3.11), we can write

$$\left|\frac{\gamma}{2}\left(D\left[\frac{3}{2}((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - \frac{1}{2}((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2)\right], \delta\frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}\right)\right|$$
$$= \frac{|\gamma|}{8}\delta\,|(3((\mathbf{V}^n)^2 - (\mathbf{W}^n)^2) - ((\mathbf{V}^{n-1})^2 - (\mathbf{W}^{n-1})^2), D(\mathbf{E}^{n+1} + \mathbf{E}^n))|$$
$$\le \frac{3|\gamma|}{8}\delta\|\mathbf{V}^n + \mathbf{W}^n\|_\infty\|\mathbf{E}^n\|\,\|D(\mathbf{E}^{n+1} + \mathbf{E}^n)\|$$

$$+\frac{|\gamma|}{8}\delta\|\mathbf{V}^{n-1}+\mathbf{W}^{n-1}\|_\infty\|\mathbf{E}^{n-1}\|\,\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|$$

$$\leq \frac{3|\gamma|}{8}R\delta\left(\eta\|\mathbf{E}^n\|^2+\frac{1}{2\eta}\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|^2\right)$$

$$+\frac{|\gamma|}{8}R\delta\left(\eta\|\mathbf{E}^{n-1}\|^2+\frac{1}{2\eta}\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|^2\right)$$

$$=\frac{3|\gamma|}{8}R\delta\eta\|\mathbf{E}^n\|^2+\frac{|\gamma|}{8}R\delta\eta\|\mathbf{E}^{n-1}\|^2+\frac{|\gamma|}{4\eta}R\delta\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|^2,$$

where η is an arbitrary positive constant.

Finally, the last two terms in the right of (3.9) may be bounded by means of the techniques employed for the β and γ terms. In this way, after recalling the equivalence between the L^2 and energy norms, (3.9) can be transformed into

$$\frac{1}{2k}(\|\mathbf{E}^{n+1}\|_E^2-\|\mathbf{E}^n\|_E^2)+\frac{\delta}{4}\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|^2\leq C(\|\mathbf{E}^{n+1}\|_E^2+\|\mathbf{E}^n\|_E^2+\|\mathbf{E}^{n-1}\|_E^2)$$

$$+\frac{|\gamma|}{4\eta}R\delta\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|^2+\frac{\varepsilon\delta}{2\eta}\|D(\mathbf{E}^{n+1}+\mathbf{E}^n)\|^2+\|\mathbf{L}^{n+1}\|_E^2,$$

where C is a constant depending only on the allowed parameters. On choosing η so that $\eta\geq(2\varepsilon+|\gamma|R)$, we obtain

$$\frac{1}{2k}(\|\mathbf{E}^{n+1}\|_E^2-\|\mathbf{E}^n\|_E^2)\leq C(\|\mathbf{E}^{n+1}\|_E^2+\|\mathbf{E}^n\|_E^2+\|\mathbf{E}^{n-1}\|_E^2)+\|\mathbf{L}^{n+1}\|_E^2,$$

$$1\leq n\leq N-1,$$

and Gronwall's lemma leads to (3.6).

Consistency and convergence

By definition, the truncation errors $\mathfrak{I}^n$, $n=2,\ldots,N$ are the residuals associated with the grid restrictions $\mathbf{u}^n$ of the theoretical solution. Namely (cf. (3.4)–(3.5))

$$\mathfrak{I}^{n+1}=\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{k}+D^3\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}+\beta D\left(\frac{3}{2}(\mathbf{u}^n)^2-\frac{1}{2}(\mathbf{u}^{n-1})^2\right)$$

$$+\frac{\gamma}{2}D^2\left(\frac{3}{2}(\mathbf{u}^n)^2-\frac{1}{2}(\mathbf{u}^{n-1})^2\right)+\varepsilon D^2\left(\frac{3}{2}\mathbf{u}^n-\frac{1}{2}\mathbf{u}^{n-1}\right)-\delta D\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{k},$$

$$n=1,\ldots,N-1.$$

Taylor expansion with respect to t along with standard bounds for the spatial error arising from pseudospectral differencing (see e.g. [12, lemma 2.2]) yield a bound

$$\max_{2\leq n\leq N}\|\mathfrak{I}^n\|_E=O(k^2+h^s),\qquad k,h\to 0, \tag{3.12}$$

provided that $u\in C^3([0,T],H^3)\cap C^1([0,T],H^{s+1})\cap C([0,T],H^{s+3})$ and $u^2\in$

$C([0, T], H^{s+2})$. It is important to note that the exponent s in (3.12) (i.e. the order of consistency in space) depends only on the smoothness of u. In particular, if u is indefinitely differentiable, the truncation error tends to 0 faster than any power of h. Furthermore the truncation error may even be exponentially small [12]. We can now prove:

Theorem 3.2 (*Convergence*). *Assume that: (i) The problem* (1.1)–(1.3) *possesses a unique classical solution u for which* (3.12) *holds with* $s > 1/2$. *(ii) The starting vectors satisfy* $\|\mathbf{U}^0 - \mathbf{u}^0\|_E = O(k^2 + h^s)$, $\|\mathbf{U}^1 - \mathbf{u}^1\|_E = O(k^2 + h^s)$, $k, h \to 0$. *(iii) The grids are refined in such a way that* $k = o(h^{1/4})$, $h \to 0$. *Then*

$$\max_{0 \le n \le N} \|\mathbf{U}^n - \mathbf{u}^n\|_E = O(k^2 + h^s), \qquad k, h \to 0.$$

Proof. We apply the stability bound (3.6) with $\mathbf{V}^n = \mathbf{U}^n$, $\mathbf{W}^n = \mathbf{u}^n$, $n \ge 2$. The fact that $\mathbf{U}^n$ satisfies the threshold condition (3.4) for h, k small enough is a consequence of the main theorem of [8], [9], because the order $O(k^2 + h^s)$ of consistency is larger than the order $O(h^{1/2})$ of the thresholds.

4. Numerical Example

As in [3] and [1], we consider the parameter values $\beta = -0.45000$, $\gamma = 0.37947$, $\delta = 0.04216$, $\varepsilon = 0.09487$ and the initial condition $q(x) = 0.1\sin(x)$.

The missing starting level $n = 1$ in (2.4)–(2.5) was computed by a step of the standard forward Euler scheme applied to (2.3), so that the overall algorithm is second order accurate in time. Table 1 gives the absolute errors at $x = \pi$, $t = 20$ (the true solution is $u(\pi, 20) = -0.258126$). Note that the errors in the first column are roughly independent of k. This shows that for $2J = 4$ the spatial errors dominate. For $2J = 8$, the picture is reversed and the errors show an $O(k^2)$ behaviour, so that the spatial error is negligible. Such a drastic error reduction when doubling the number of grid points is typical of pseudospectral methods and cannot be found when using finite differences or finite elements.

Table 1

	$2J = 4$	$2J = 8$
$k = 0.1$	0.005827	0.000794
$k = 0.05$	0.006215	0.000183
$k = 0.025$	0.006295	0.000047
$k = 0.0125$	0.006315	0.000012

As a comparison we have implemented the method suggested in [1], with the Adams-Bashforth/trapezoidal rule time-stepping used in (2.4). The errors are shown in Table 2. Errors corresponding to smaller values of k are not given, as a decrease in k increases the computational costs without reducing the error. It is clear that the pseudospectral scheme is much more accurate. As far as computational costs

Table 2

	$2J = 32$	$2J = 64$	$2J = 128$
$k = 0.1$	0.030588	0.006865	0.001118

go, the most expensive run in Table 1 ($2J = 8$, $k = 0.0125$) took 7 seconds CPU time in a VAX 11/760 with the Fast Fourier Transform coded by us in FORTRAN. The most expensive run in Table 2 took 13 seconds. (The CPU time of the remaining runs can be found from those we have just given, as the cost increases linearly with J and $1/k$.) When both accuracy and cost are taken into account the superiority of the pseudospectral method is perfectly clear.

Acknowledgement

This work is part of the Project PB-86-0313 supported by "Fondo Nacional para el Desarrollo dela Investigación Científica y Técnica".

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