# Foreword

This book is devoted to a subject, the numerical solution of ordinary differential equations, where practical relevance meets mathematical beauty in a unique way. It is masterfully written, as befits its author, someone whose past and current contributions to the field are second to none in history.

The numerical integration of differential equations plays a crucial role in all applications of mathematics. Virtually all the scientific laws that govern the physical world are differential equations; therefore making explicit the implications and consequences of the laws requires finding the solutions to the equations. This necessity, coupled with the unfortunate fact that there is no general rule to solve analytically a given differential equation, has led over the years to the introduction by the best mathematical minds of a raft of techniques applicable only in particular equations or oriented to specific features of the solutions sought. While some of those efforts have significantly spurred the development of mathematics in the last three hundred years (they have given us, for example, the theory of special functions, Lie groups and topology), numerical methods are the only master key to solve differential equations.

The subject matter of this volume is not only of exceptional relevance due to its importance in practical applications; it constitutes a rich and elegant branch of mathematics with exceptionally distinguished roots. As is well known, the simplest numerical algorithm to solve (ordinary) differential equations was suggested by Euler in the mid eighteenth century. It is less known that, for Euler, what we now call Euler's method was just a stepping stone in his insightful presentation of the Taylor series method of arbitrary order. Euler also discussed with care the use of variable order and variable step lengths and implementation details. The next milestone of the subject, the introduction of multistep algorithms, was laid down in the mid nineteenth century by Adams, the scientist best known for having co-discovered the existence of the planet Neptune using only mathematics. Another important class of numerical integrators was introduced by Runge and systematized by Kutta around the year 1900. Thus, one hundred years ago, the sciences had a pressing need to solve differential equations, the mathematicians had put forward many useful algorithms to solve them ... and yet there was a big gap: carrying out the required computations was typically unfeasible when pencil and paper or mechanical machines were the only ways of performing arithmetic operations. It is not an exaggeration to claim, that the need to implement in practice the integration algorithms of Adams, Runge and Kutta led to the conception and construction of (digital, electronic) computers;

after all, one the first computers was named ENIAC, Electronic Numerical Integrator and Computer. Since then, computers have of course revolutionized not only the mathematical solution of differential equations but almost everything else.

It was only natural that when the use of computers became widespread, mathematicians asked themselves whether the venerable integrators introduced by Adams, Runge and Kutta were the best conceivable. As it turned out, in the multistep field, the beautiful mathematics of Dahlquist showed that for nonstiff problems it is not really feasible to do much better than Adams had suggested: while, by the addition of extra free parameters, it is possible to concoct more sophisticated integrators, these are doomed to be unstable. In the Runge-Kutta realm the situation is just the opposite; the many degrees of freedom in the world of Runge-Kutta integrators have revealed themselves capable of providing a good integrator for each situation.

The construction and analysis of specific Runge-Kutta schemes is a daunting job if one approaches it as Runge and Kutta did; these schemes are highly nonlinear with a remarkable Matrioshka doll structure, where the vector field has to be evaluated at an expression that involves the vector field evaluated at an expression that involves the vector field . . . . Mathematics owes to the author of this book a simple and elegant, alternative, general methodology based on the use of trees and other algebraic ideas. It is thanks to J. C. Butcher's techniques that many authors have been able in the last decades to develop Runge-Kutta methods tailored to different needs and to implement them in useful software. Many such techniques have found uses away from numerical mathematics in fields such as quantum field theory and noncommutative geometry. Connes (Field medallist 1982) and Kreimer, writing on renormalization theories, state: "We regard Butcher's work on the classification of numerical integration methods as an impressive example that concrete problem-oriented work can lead to far-reaching conceptual results."

The author wrote an earlier text restricted to Runge-Kutta and general linear methods. This is the third, significantly updated, edition of the present book; I am sure it will be as well received as the two preceding editions.

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