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An Extension of the Lax-Richtmyer Theory

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Summary. We give a rigorous proof of the validity of the Lax equivalence theorem when the true solution at time t and its approximation lie in different spaces, thus modelling the practical situation where the former is a function of the space variables and the latter is only defined at grid points.

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1. Introduction

The celebrated Lax-Richtmyer theory [4, 7] investigates the discretization of properly posed Cauchy problems

$$\begin{aligned} du/dt &= Au \qquad 0 \leqq t \leqq T \\ u(0) &= u_0 \end{aligned} \tag{1}$$

by means of a stepping procedure

$$u^{n+1} = C(\Delta t)\, u^n,$$

where the elements $u(t)$, $0 \leqq t \leqq T$ and the approximations u^n, $n = 0, 1, \ldots, [T/\Delta t]$, lie in *the same* Banach space X, and A, $C(\Delta t)$ are linear operators mapping X into X. This set-up is, in a sense, only an idealization [5, p. 68] of the procedures followed in practice, where the element $u(t)$ is a function of the space variables, while u^n is only defined as a grid function. Attemps to fill this gap, without altering the framework of the Lax-Richtmyer theory, are usually made either by interpolating u^n or by pretending that the difference equations hold for all values of the space variables [7, p. 30]. The applicability of the second device is very restricted, while the first one suffers from the arbitrariness of the interpolation procedure [10, p. 7]. On the other hand, general frameworks for the study of *discretizations* have appeared in the literature (see [10] among many others). Those frameworks, while catering for problems more general than (1), typically restrict their scope to the implication "stability and consistency $\Rightarrow$ convergence", i.e. the part of the Lax equivalence theorem which can be dealt with by elementary means.

The purpose of this article is to provide a rigorous proof of the validity of the Lax-Richtmyer Theory when the approximations u^n are allowed to lie in normed spaces $X_{\Delta t}$ that vary with Δt (cf. Sect. 3).

Section 2 is devoted to an abstract exposition of our results. The practical implications are considered in Sect. 3

The reader is referred to [1] for other extensions of the Lax-Richtmyer theory, and to [6] for a discussion of the practical relevance of the functional analysis ideas involved in the theory and its extensions.

For simplicity we shall use the notation h rather than Δt.

2. The Main Result

We consider the initial value problem (1), where A is a linear operator $A: D(A) \subset X \to X$, (X a Banach space with norm $\|.\|$). It is assumed that the problem is properly posed [5, 7], so that in particular there is a uniformly bounded family of linear operators $E(t)$, $0 \leqq t \leqq T$ such that, for $u_0 \in X$ the mapping $t \to E(t) u_0$ is the unique (generalized) solution of (1).

Definition 1. A difference scheme is a family of pairs $(X_h, C(h))_{h \in H}$, where:

i) The set H of indices h is a subset of the set of positive real numbers and 0 is an accumulation point of H.

ii) X_h are normed spaces with norms $\|.\|_h$.

iii) $C(h)$ are continuous linear operators in X_h.

iv) For each compact subset K of H there exists a positive constant C such that $\|C(h)\|_h \leqq C$ for h in K. (Here $\|.\|_h$ denotes the operator norm in X_h).

Definition 2. A difference scheme is said to be stable (in $[0, T]$), if there exists a positive constant C, such that for any h in H and any positive integer n with $nh \leqq T$, the bound

$$\|C^n(h)\|_h \leqq C \tag{2}$$

holds.

Note that neither the notion of difference scheme nor that of stability relate to the problem (1).

Definition 3. A consistent difference method for the initial value problem (1) is a family of triplets $(X_h, C(h), r_h)_{h \in H}$, where $(X_h, C(h))_{h \in H}$ is a difference scheme and

i) r_h are continuous linear operators mapping X onto X_h, such that

$$\sup_h \|r_h\|_{L(X, X_h)} < \infty, \tag{3}$$

and a constant R can be found such that for each $v \in X_h$, with $\|v\|_h \leqq 1$, there exists $w \in X$ satisfying $\|w\| \leqq R$, $v = r_h w$.

ii) There exists a dense subset $Y \subset X$, such that for $u_0 \in Y$, and uniformly in $0 \leqq t \leqq T$,

$$\lim_{\substack{h \to 0 \\ h \in H}} \|h^{-1}(r_h E(t+h) u_0 - C(h) r_h E(t) u_0)\|_h = 0.$$

The norm in (3) is of course the operator norm derived from the norms in X and X_h.

Definition 4. A consistent difference method for (1) is convergent if for each $u_0 \in X$ and each t, $0 \leqq t \leqq T$

$$\lim_{j \to \infty} \|r_{h_j} E(t) u_0 - C(h_j)^{n_j} r_{h_j} u_0\|_{h_j} = 0, \tag{4}$$

where (h_j) is an arbitrary sequence with $h_j \in H$, $\lim h_j = 0$ and (n_j) is an arbitrary sequence of integers with $0 \leqq n_j h_j \leqq T$, $\lim n_j h_j = t$.

Theorem. *A consistent difference method for* (1) *is convergent if and only if its difference scheme is stable in* $[0, T]$.

Proof. The implication stable→convergent is proved exactly as in the Lax-Richtmyer theory [7]. In order to prove that stability is necessary for convergence, we need the following extension of the uniform boundedness principle.

Lemma. *Let X be a Banach space,* $(X_h)_{h \in H}$ *a family of normed spaces,* $T_h: X \to X_h$ *linear operators. If for each* $x \in X$, $\sup \|T_h x\|_h < \infty$, *then* $\sup \|T_h\|_{L(X, X_h)} < \infty$.

This lemma, to our best knowledge, is not available in the literature. However it is readily proved by means of an adaptation of any of the usual proofs of the standard Banach-Steinhaus theorem. In particular the proof of Theorem 1 of [3] applies (almost) verbatim.

We are now ready to study the necessity of the stability. First, we note that if the method is convergent then, for each $u_0 \in X$

$$\sup \|C(h)^n r_h u_0\|_h < \infty$$

where the supremun is taken for $h \in H$, n integer, $0 \leqq nh \leqq T$, (see [7, p. 46]). From the lemma, we conclude that

$$\sup \|C(h)^n r_h\|_{L(X, X_h)} = A < \infty.$$

Now if $v \in X_h$, $\|v\|_h \leqq 1$, then by hypothesis, $v = r_h w$ with $\|w\| \leqq R$ and therefore

$$\|C(h)^n v\|_h = \|C(h)^n r_h w\|_h \leqq A \|w\| \leqq AR$$

so that the scheme is stable.

3. Discussion

In practical applications X is a space of functions of one or several "space" variables. It is assumed that the parameters which govern the space discretization [9] (mesh size, element diameter etc.) have been expressed as functions of $h = \Delta t$. In finite differences the spaces X_h consist of grid functions. We have not assumed that the X_h are finite dimensional, thus catering for the possibility of grids with an infinite number of points. (These may arise in pure initial value problems in PDEs.) When the discretization in space is performed by means of finite elements, the X_h are subspaces of X. Note that by taking $X = X_h$ we recover the Lax-Richtmyer theory.

We have chosen to make a distinction between a difference *scheme* and a difference *method*. Only the former is required in order to produce the numerical solution. The scheme does not refer to any specific initial value problem. Stability is a property of the scheme. On the other hand a *method* is, in agreement with the etymology of the word, a way of (approximately) solving a given problem. A method consists of a scheme $(X_h, X(h))$ and a family of mappings r_h which make it possible to compare the true solution $u(t)$ with the numerical solution provided by the scheme. In fact, one scheme may give rise to different methods, as studied in [2].

In order to measure the *global* error, i.e. to compare an element $u(t)$ in X with an element u^n in X_h, one can adopt one of the two following procedures [10]:

i) To map X into X_h by a "restriction" operator r_h as in Sect. 2. In a finite difference setting this means to compare the numerical solution with a suitable restriction to the grid of the true solution. For finite elements r_h would be a least-squares or Galerkin projection of X onto the subspace X_h.

ii) To map X_h into X by a prolongation operator p_h. For finite differences this would mean interpolation of the grid values.

For finite differences i) seems far more natural, as ii) contains a large amount of arbitrariness. For finite elements $X_h \subset X$, and the identity (or injection) appears as an obvious, natural candidate for prolongation. However, in this case we can write $u(t) - p_h u^n = u(t) - u^n = (u(t) - r_h u(t)) + (r_h u(t) - u^n)$. Thus, the error $u(t) - p_h u^n$ associated with alternative ii) equals the error $r_h u(t) - u^n$ in alternative i), plus a term $u(t) - r_h u(t)$ which merely reflects the approximation properties of X_h and does not depend on the scheme [9].

We conclude that alternative i), as used in Sect. 2, is more natural than alternative ii) (see [10]).

We also believe that the definitions of consistency, stability and convergence employed here are those one would expect. In fact, they have been used in the text [5], where the implication stability→convergence is proved. Furthermore our definitions fit the general framework of [10].

The hypotheses made on r_h in Definition 3 hold in all practical applications: for instance if X is a Hilbert space and r_h an orthogonal projection, or if X is a space of continuous functions with the sup norm and r_h is a point restriction to a grid, etc. In the case of L^p spaces, whose elements are only defined almost everywhere, point restrictions do not make mathematically sense and must be replaced by averages of cell values. (Note that from a practical point of view [2] and in cases of steep spatial gradients, it is not advisable to considerer the numbers produced by the scheme as approximations to point values of the true solution.)

We also emphasize that if X were an L^p space and if it were wanted to use point restrictions for the role of r_h in (4), then, in Definition 4, the requirement "for each $u_0 \in X$" would have to be weakened into "for each u_0 in a dense subset of X", since point restrictions are only densely defined (namely, they are only defined for continuous functions). Now the implication "convergence ⇒ stability" does *not* hold after such modification of the definition of convergence. (See [6] for a detailed discussion of this point.)

In order to deal with point restrictions and similar situations the following proposition may be useful. (The proof is obvious, using the bound (2).)

Proposition. *Let* $(X_h, C(h), r_h)_{h\in H}$ *a consistent, convergent difference method for* (1) *and let* $(\tilde{r}_h)_{h\in H}$ *be a family of linear operators from a common domain* $Z\subset X$ *into* X_h. *Assume that for each* v *in* Z

$$\lim_{h\to 0} \|r_h v - \tilde{r}_h v\|_h = 0. \tag{5}$$

Then, if $u_0\in Z$, $t\in[0, T]$, $E(t)u_0\in Z$, *it follows that*

$$\lim_{j\to\infty} \|\tilde{r}_{h_j} E(t) u_0 - C(h_j)^{n_j} \tilde{r}_{h_j} u_0\|_{h_j} = 0 \tag{6}$$

for each sequence (h_j) *with* $h_j\in H$, $\lim h_j = 0$ *and each sequence of integers* (n_j) *with* $0\leqq n_j h_j \leqq T$, $\lim n_j h_j = t$.

An example of the application of the proposition will now be given. Let $X = L^p(0,1)$, $1\leqq p\leqq\infty$. The interval $(0,1)$ is partitioned by a uniform grid $0 = x_0 < x_1 < \ldots < x_{N(h)} = 1$ with diameter h. We take $X_h = R^{N(h)+1}$ with the discrete L^p norm and chose for r_h (respectively $\tilde{r}_h$) the restrictions based on element averages (respectively nodal values). Then (5) holds for $Z = C[0,1]$. The proposition shows that (6) holds for each choice of $(C(h))$ leading to a convergent method and for each u_0 and t such that u_0, $E(t)u_0$ are continuous (i.e. for each u_0, t such that $\tilde{r}_h u_0$, $\tilde{r}_h E(t) u_0$ make sense).

References

1. Ansorge, R.: Differenzapproximationen partieller Anfangswertaufgaben. Stuttgart: B.G. Teubner 1978
2. Cullen, M.J.P., Morton, K.W.: Analysis of evolutionary error in Finite Element and other methods. J. Computational Phys. **34**, 245–267 (1980)
3. Kantorovich, L.V., Akilov, G.P.: Functional Analysis, 2nd Ed. Oxford: Pergamon 1982
4. Lax, P.D., Richtmyer, R.D.: Survey of stability of linear finite difference equations. Comm. Pure Appl. Math. **9**, 267–293 (1956)
5. Meis, T., Markowitz, V.: Numerical solution of partial differential equations. New York: Springer 1981
6. Palencia, C., Sanz-Serna, J.M.: Equivalence theorems for incomplete spaces: An appraisal. IMA J. Numer. Anal. **4** (in press)
7. Richtmyer, R.D., Morton, K.W.: Difference methods for initial value problems. New York: Interscience 1967
8. Rosinger, E.E.: Stability and convergence for non-linear difference schemes are equivalent. J. Inst. Math. Appl. **26**, 143–149 (1980)
9. Sanz-Serna, J.M.: Convergent approximations to partial differential equations and stability concepts of methods for stiff systems of ordinary differential equations. Preprint
10. Stetter, H.J.: Analysis of discretization methods for ordinary differential equations. Berlin, Heidelberg, New York: Springer 1973

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