Soliton and antisoliton interactions in the "good" Boussinesq equation

V. S. Manoranjan
Department of Chemical and Process Engineering, University of Surrey, Guilford, Surrey GU2 5HX, England

T. Ortega and J. M. Sanz-Serna
Departamento de Matemática Aplicada y Computación, Facultad de Ciencias, Universidad de Valladolid, Valladolid, Spain

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The solitary-wave interaction mechanism for the good Boussinesq equation is investigated and found to be far more complicated than was previously thought. Three salient features are that solitary waves only exist for a finite range of velocities, that large solitons can turn into so-called antisolitons, and that it is possible for solitons to merge and split. Small solitons, however, appear to be stable. The existence of a potential well is linked to the different behaviors observed between small and large initial conditions.

I. INTRODUCTION

The importance of soliton-producing nonlinear wave equations is well understood among theoretical physicists and applied mathematicians. An equation that produces solitons and has received comparatively little attention in the literature is

$$u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx}. \tag{1.1}$$

This is referred to as the "good" Boussinesq equation (McKean[1]) or the nonlinear beam equation.[2] The related equation

$$u_{tt} = u_{xxxx} + u_{xx} + (u^2)_{xx},$$

known as the "bad" Boussinesq equation, has been studied by Hirota.[3] In a recent article, Manoranjan *et al.*[4] obtained a closed-form expression for the two soliton interactions of (1.1) and carried out numerical experiments to demonstrate the possibility of the breakup of an initial pulse into two solitons.

In this paper we show that the interaction mechanism is more complicated than that reported in Ref. 4. It turns out that when small amplitude solitons of (1.1) collide, they emerge from the nonlinear interaction with no change in shape or velocity. However, the large amplitude solitons change into so-called antisolitons as they come out from the interaction. We show that this difference in behavior is linked to the existence of a potential well for (1.1). Further, the existence of a local minimum for the potential energy enables us to investigate the existence of a class of solutions that remain bounded for all time, along with another class of solutions that blow up in finite time.

Throughout the paper our attention is confined to real-valued solutions u of (1.1) defined in $-\infty < x < \infty$.

II. PRELIMINARIES

A. Conservation laws

It is clear that if u is a smooth solution of (1.1) that vanishes, along with its derivatives, as $|x| \to \infty$, then the quantity

$$J(u) = \int_{-\infty}^{\infty} u_t \, dx$$

is an invariant of motion, whereas

$$I(u) = \int_{-\infty}^{\infty} u \, dx \tag{2.1}$$

varies in time as $I = Jt + \text{const}$. Since we are interested in solutions that remain bounded as t increases, we restrict our attention to initial conditions that satisfy $J = 0$. It is then expedient to introduce a new function w defined by

$$w(x,t) = \int_{-\infty}^{x} u_t(\xi,t) d\xi,$$

which also vanishes with its derivatives as $|x| \to \infty$. In terms of the functions u,w, the equation (1.1) becomes

$$w_t = -u_{xxx} + u_x + (u^2)_x, \quad u_t = w_x, \tag{2.2}$$

a system that conserves the functional I in (2.1) and the functionals M and E given by

$$M(w,u) = \int_{-\infty}^{\infty} wu \, dx, \tag{2.3}$$

and

$$E(w,u) = T(w) + V(u), \tag{2.4}$$

with

$$T(w) = \frac{1}{2} \int_{-\infty}^{\infty} w^2 \, dx, \tag{2.5}$$

and

$$V(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2} u_x^2 + \frac{1}{2} u^2 + \frac{1}{3} u^3 \right) dx. \tag{2.6}$$

Note that the system (2.2) is in Hamiltonian form and that E,T,V act as total energy, kinetic energy, and potential energy, respectively.

B. Solitons and antisolitons

We now look for traveling wave solutions of (2.2) of the form $w = w(\xi)$, $u = u(\xi)$, $\xi = x - ct$. With a prime denoting differentiation with respect to ξ, the system (2.2) then reads

$$-cw' = -u''' + (u^2)' + u', \quad -cu' = w'. \tag{2.7}$$

Elimination of w', followed by an integration in which the

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integration constant must be zero in view of the boundary conditions for u, leads to

$$-c^2u = u'' - u^2 - u. \tag{2.8}$$

If $c^2 > 1$, then the origin $u = 0$, $u' = 0$ is a center in the phase plane of (2.8), and hence nontrivial solutions such that $u,u' \to 0$ as $|\xi| \to \infty$ are not possible. On the other hand, when $c^2 < 1$, the origin is a saddle point as depicted in the phase plane shown in Fig. 1, and the homoclinic trajectory $O \to A \to B \to O$ represents a soliton. The possible velocities $-1 < c < 1$ can be described in the form $c = \varepsilon(1 - P^2)^{1/2}$, where P is a parameter $0 < P \leqslant 1$, and $\varepsilon = +1$ or -1 determines whether the wave moves to the right or to the left. A simple integration in (2.8) yields the analytic form of the soliton as

$$u(\xi) = (-3P^2/2)\operatorname{sech}^2[(P/2)(\xi - \xi_0)]. \tag{2.9}$$

The (real) integration constant ξ_0 gives the initial location of the wave. It should be noted that the velocity c of the soliton is related to the amplitude $A = 3P^2/2$ by $A = \frac{3}{2}(1 - c^2)$. For the rightward- and leftward-traveling soliton of parameter P, the quantities in (2.1) and (2.3)–(2.6) are given by

$$I_{P,\varepsilon} = -6P, \tag{2.10}$$

$$M_{P,\varepsilon} = -6\varepsilon(1 - P^2)^{1/2}P^3, \tag{2.11}$$

$$E_{P,\varepsilon} = \tfrac{6}{5}P^3(5 - 4P^2), \tag{2.12}$$

$$T_{P,\varepsilon} = 3P^3(1 - P^2), \tag{2.13}$$

$$V_{P,\varepsilon} = \tfrac{3}{5}P^3(5 - 3P^2). \tag{2.14}$$

It should also be pointed out that the functions $w(\xi), u(\xi)$ corresponding to the soliton solve the variational problem

$$\delta E(w,u) = 0, \quad \text{subject to } M(w,u) = M_{P,\varepsilon}. \tag{2.15}$$

This can be verified by comparing the Euler–Lagrange equations of (2.15) with the system (2.7) satisfied by solitons.

Returning to the phase plane in Fig. 1 ($c^2 < 1$), and if we allow singular solutions, the trajectory $O \to C \to \infty \to D \to O$ also provides a traveling wave with velocity c, whose analytic form is found to be

$$u(\xi) = (3P^2/2)\operatorname{cosech}^2[(P/2)(\xi - \xi_0)]. \tag{2.16}$$

This singular solution, which has a double pole at $\xi = \xi_0$, will be referred to as antisoliton. It is helpful to combine Eqs. (2.9) and (2.16) into a single form,

$$u(\xi) = -6P^2e^{\eta}(1 + e^{\eta})^{-2}, \quad \eta = P(\xi - \xi_0) + i\pi\sigma, \tag{2.17}$$

where $i^2 = -1$, and $\sigma = 0$ gives the soliton and $\sigma = 1$ the antisoliton.

The possibility of traveling waves with velocities $c = +1$ or -1 has not been discussed so far. In these cases, a study of (2.8) reveals that there is a singular solution given by $u = 6(\xi - \xi_0)^{-2}$, but no solution of the soliton type exists. The singular solution just mentioned can also be obtained by taking in (2.16) the limit $P \to 0$, i.e., the limit $|c| \to 1$.

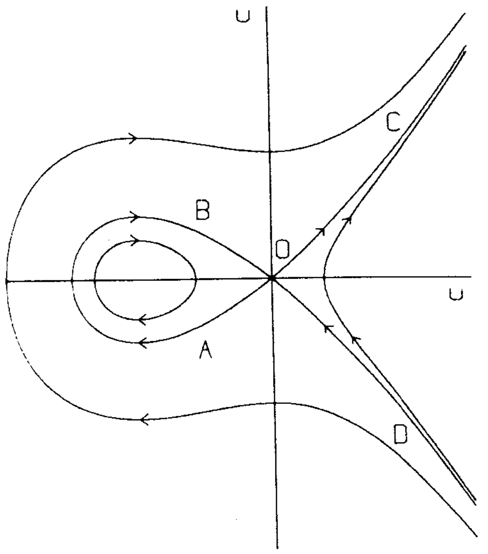

FIG. 1. Phase plane, $c^2 < 1$.

III. SOLITON AND ANTISOLITON INTERACTION

Following a technique used by Hirota,[3] Manoranjan *et al.*[4] constructed the family of solutions of (1.1) given by

$$u = -6(f_{xx}f - f_x^2)/f^2,$$
$$f(x,t) = 1 + \exp(\eta_1) + \exp(\eta_2) + a\exp(\eta_1 + \eta_2), \tag{3.1}$$

with

$$\eta_j = P_j[x - \varepsilon_j v_j t + x_j^0],$$
$$\varepsilon_j = \pm 1, \quad 0 < P_j \leqslant 1, \quad j = 1,2, \tag{3.2}$$

$$a = \frac{(\varepsilon_1 v_1 - \varepsilon_2 v_2)^2 - 3(P_1 - P_2)^2}{(\varepsilon_1 v_1 - \varepsilon_2 v_2)^2 - 3(P_1 + P_2)^2}, \tag{3.3}$$

and

$$v_j = (1 - P_j)^{1/2}, \quad j = 1,2. \tag{3.4}$$

Only real values of the phases $\eta_j, j = 1,2$, featuring in (3.2) were considered in Ref. 4, while in the present study we let η_j have the following complex form:

$$\eta_j = P_j[x - \varepsilon_j v_j t + x_j^0] + i\pi\sigma_j,$$
$$\varepsilon_j = \pm 1, \quad 0 < P_j \leqslant 1, \quad \sigma_j = 0,1, \quad j = 1,2. \tag{3.5}$$

With this choice of η_j, (3.1) is still a valid family of real-valued solutions u. We next show that (3.1) and (3.3)–(3.5) describe the exact interaction of solitons and antisolitons. To simplify matters we only discuss the case $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, but the other possibilities in the choice of ε_j, $j = 1,2$, can be analyzed in a similar fashion (see Ortega[5]).

Elimination of f in (3.1) yields

$$\frac{u}{-6} = \frac{P_1^2e^{\eta_1} + P_2^2e^{\eta_2} + [a(P_1 + P_2)^2 + (P_1 - P_2)^2]e^{\eta_1 + \eta_2} + ae^{\eta_1 + \eta_2}(e^{\eta_1}P_2^2 + e^{\eta_2}P_1^2)}{[1 + e^{\eta_1} + e^{\eta_2} + ae^{\eta_1 + \eta_2}]^2}. \tag{3.6}$$

The behavior of (3.6) depends very much on the value of the number a in (3.3). Several cases must be considered.

(i) $0<a<\infty$. This corresponds to the interior of the regions I_a, I_b, I_c in Fig. 2. More specifically, $0<a<1$ in the interior of the regions I_b, I_c, while $1<a<\infty$ in the interior of the region I_a. By arguing as in Whitham (Ref. 6, Sec. 17.2.), it is found that as $t\to-\infty$, the solution (3.6) becomes just the linear superposition of two traveling waves of the form (2.17) with

$$\eta=\eta_{1-}=P_1[x-v_1t+x_1^0]+i\pi\sigma_1,$$
$$\eta=\eta_{2-}=P_2[x+v_2t+x_2^0+P_2^{-1}\log a]+i\pi\sigma_2.$$

As $t\to\infty$, the solution consists of the traveling waves

$$\eta=\eta_{1+}=P_1[x-v_1t+x_1^0+P_1^{-1}\log a]+i\pi\sigma_1,$$
$$\eta=\eta_{2+}=P_2[x+v_2t+x_2^0]+i\pi\sigma_2.$$

Thus the right moving wave (resp. the left moving wave) emerges from the interaction having a shift in position $\Delta x=P_1^{-1}\log a$ (resp. $\Delta x=-P_2^{-1}\log a$), but with no change in shape or velocity. Note that there are four choices for σ_1, σ_2, so that the interacting waves could be both solitons, both antisolitons, or a soliton along with an antisoliton. The right-going wave is shifted to the left ($\log a>0$) for P_1,P_2 in the region I_a and to the right in the regions I_b and I_c. The shift in the left-going wave occurs in the direction opposite of that in the right-going wave, in agreement with the fact that the total change in the phases η_1,η_2 must be zero (cf. Hirota[3]).

(ii) $a<0$. (Interior of region II.) Here the behavior of (3.6) as $t\to\infty$ is given by

$$\eta=\eta_{1-}=P_1[x-v_1t+x_1^0]+i\pi\sigma_1,$$
$$\eta=\eta_{2-}=P_2[x+v_2t+x_2^0+P_2^{-1}\log|a|]+i\pi\sigma_2^*,$$

where $\sigma_2^*=1-\sigma_2$. For $t\to\infty$ we have

$$\eta=\eta_{1+}=P_1[x-v_1t+x_1^0+P_1^{-1}\log|a|]+i\pi\sigma_1^*,\quad \sigma_1^*=1-\sigma_1,$$
$$\eta=\eta_{2+}=P_2[x+v_2t+x_2^0]+i\pi\sigma_2.$$

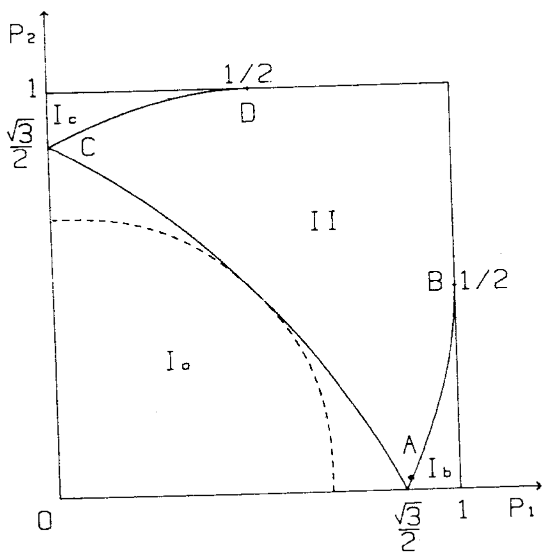

FIG. 2. Soliton and antisoliton interaction for the case $\varepsilon_1=1$, $\varepsilon_2=-1$.

It is apparent that now the waves not only undergo a shift but also change their nature; a soliton/antisoliton entering an interaction emerges as an antisoliton/soliton. Again four cases $\sigma_j=0,1$, $j=1,2$, are possible.

(iii) $a=0$. (Arcs AB and CD.) We consider first the arc AB where $P_1>P_2$. By taking limits in (3.6) along lines $x=mt+n$, m,n real constants, and comparing with (2.17), we find that as $t\to-\infty$ the solution consists of a single wave with phase

$$P_1[x-v_1t+x_1^0]+i\pi\sigma_1,$$

while for $t\to\infty$ there are two waves,

$$P_2[x-v_2t+x_2^0]+i\pi\sigma_2,$$
$$P_3[x-v_3t+P_3^{-1}(P_1x_1^0-P_2x_2^0)]+i\pi\sigma_3,$$

with

$$P_3=P_1-P_2,\quad v_3=(1-P_3^2)^{1/2},$$
$$\sigma_3=\sigma_1+\sigma_2(\mathrm{mod}\ 2).$$

In obtaining the last outcoming wave, use must be made of the relation $(P_1-P_2)[1-(P_1-P_2)^2]^{1/2}=P_1v_1+P_2v_2$, which follows from $a=0$ in (3.3). These formulas mean that a single soliton ($\sigma_1=0$) can split into either two solitons ($\sigma_2=0$) or two antisolitons ($\sigma_2=1$). The outgoing waves move in opposite directions and the corresponding parameters satisfy $P_2+P_3=P_1$ [cf. (2.10) and the conservation of (2.1)]. On the other hand, an incoming antisoliton ($\sigma_1=1$) splits into either a left-going soliton and a right-going antisoliton or into a left-going antisoliton and a right-going soliton. It is perhaps useful to observe that if we represent by S and A the soliton and antisoliton, respectively, then the rules

$$S\to S+S,\quad S\to A+A,\quad A\to S+A,\quad A\to A+S,$$

familiar from Boolean algebra, govern the possible interactions.

The arc CD, where $P_2>P_1$, contains the merging of two incoming waves with parameters P_1, P_2-P_1 into a single wave with parameter P_2. The possible interactions can be represented as

$$S+S\to S,\quad A+A\to S,\quad S+A\to A,\quad A+S\to A.$$

(iv) $a=\infty$. (Arc AC.) This case corresponds either to waves of parameters P_1,P_2 merging into a single wave of parameter P_1+P_2, or to the splitting of a single wave with parameter P_1+P_2 into two waves with parameters P_1,P_2. The corresponding analysis can be performed by taking limits in (3.6) for $a\uparrow\infty$, or $a\downarrow-\infty$. However, taking limits directly in (3.6) results in the trivial solution $u=0$, and it is therefore necessary to make the parameters x_1^0 and/or x_2^0 functions of a before letting $|a|\to\infty$. For instance, we could take η_1 as given in (3.5) (x_1^0 a fixed constant) and

$$\eta_2=P_2[x+v_2t+m+P_2^{-1}\log|a|]+i\pi\sigma_2,\quad m\ \text{constant}.$$

Although we have only considered two-wave interactions, N-wave interactions with a far more complicated dynamics are also possible and can be studied by using a formula analogous to that of Hirota.[3]

IV. THE POTENTIAL WELL

In the preceding sections we have allowed singular solutions, i.e., solutions with poles. For instance, the $S \to A + A$ splitting solution possesses two poles past the interaction time. In the rest of the paper we exclude this possibility and say that a solution ceases to exist at the time when poles develop. According to this point of view, standard in mathematical analysis, it has been shown in Sec. III that some of the solutions of (2.2) exist for all times whereas other solutions cease to exist at a finite value of t. This difference in behavior, numerically verified in Ref. 4, is explained in the next section in terms of a potential well studied below.

If v is a function of x, $-\infty < x < \infty$, we define the potential energy $V(v)$ as in formula (2.6). According to Sobolev's imbedding theorem (Adams[7]), the expression for $V(v)$ makes sense whenever v is in H^1, i.e., whenever v is square integrable with a square-integrable distributional derivative. It is clear that V can take arbitrarily large positive and negative values. The stationary points of the functional V are easily found to be given by the functions $v_1 = 0$ and $v_{2,b} = -\frac{3}{2}\mathrm{sech}^2[(x-b)/2]$, with b an arbitrary real constant. Note that in view of (2.9), the function $v_{2,b}$ provides the shape of the soliton with parameter $P = 1$, amplitude $A = \frac{3}{2}$, and velocity $c = 0$. This is in agreement with the fact that the functions for which the gradient of the potential vanishes give rise to time-independent solutions of the Hamilton equations (2.2).

The function $v_1 = 0$ provides a local minimum for the functional V, because the Sobolev inequality

$$\left|\int_{-\infty}^{\infty} v^3\,dx\right| \leqslant \int_{-\infty}^{\infty} |v|^3\,dx = \|v\|_{L^3}^3 \leqslant K^3 \|v\|_{H^1}^3 = K^3\left(\int_{-\infty}^{\infty} (v^2 + v_x^2)\,dx\right)^{3/2} \qquad (4.1)$$

(K is the imbedding constant) reveals that the cubic term in the potential is negligible in the H^1 neighborhood of $v_1 = 0$.

To study the behavior of V near $v = v_{2,b}$, we make the change $v = v^* + v_{2,b}$. The functional $V(v^* + v_{2,b}) - V(v_{2,b})$ is of the form $Q(v^*) + C(v^*)$, where

$$Q(v^*) = \int_{-\infty}^{\infty}\left[\frac{1}{2}v_x^{*2} + \frac{1}{2}v^{*2} - \frac{3}{2}\mathrm{sech}^2\frac{x-b}{2}\,v^{*2}\right]dx,$$

and C is cubic in v^*. The spectrum of Q is known in closed form (e.g., Whitham,[6] Sec. 17.5), and it turns out that the functional is indefinite, hence $v_{2,b}$ is a saddle point of the potential V.

The depth d of the potential well around the local minimum v_1 is defined by

$$d = \inf\left\{V(v): v \text{ in } H^1,\ v \neq 0,\ \int_{-\infty}^{\infty}[v_x^2 + v^2 + v^3]\,dx = 0\right\}. \qquad (4.2)$$

Note that $d > 0$, because if $v \neq 0$ satisfies the integral constraint in (4.2), then

$$V(v) = \frac{1}{6}\|v\|_{H^1} = -\frac{1}{6}\int_{-\infty}^{\infty} v^3\,dx,$$

which, in view of (4.1), implies that $\|v\|_{H^1}$ [and hence $V(v)$] is bounded away from zero. If we denote the potential well by W, then W consists of functions v with potential energy below the depth d and satisfying the condition

$$-\int_{-\infty}^{\infty}[v_x^2 + v^2 + v^3]\,dx \leqslant 0. \qquad (4.3)$$

When v possesses a square-integrable second derivative, an integration by parts of the v_x^2 term in (4.3) shows that this condition can be interpreted as the requirement that the angle (in the sense of the L^2 metric) between the function v and the force density $\delta V(v)/\delta v$ be obtuse. In symbols W is defined by

$$W = \{v \text{ in } H^1\colon V(v) < d \text{ and (4.3) holds}\}.$$

In view of (4.1), W is a neighborhood of the origin in the space H^1. Furthermore, for v in W,

$$\|v\|_{H^1}^2 = \int_{-\infty}^{\infty}[v_x^2 + v^2]\,dx = 6V(v) - 3\int_{-\infty}^{\infty}[v_x^2 + v^2 + v^3]\,dx \leqslant 6d, \qquad (4.4)$$

so that W is bounded. The key issue is that if $w(t),u(t)$ is a solution of the system (2.2) that is smooth enough to conserve the energy $E(w,u)$ in (2.4)–(2.6), and such that initially $E(t=0) < d$ and $u(t=0)$ is in W, then $u(t)$ remains in W for all later times. In fact, if u were to leave the well at a time t_0, then at that time (4.3) would hold with $=$ instead of $\leqslant$, which, since E is below d, is in contradiction with (4.2).

The Euler–Lagrange equation for the constrained minimization problem in (4.2) is easily shown to have no solution other than the function $v_{2,b}$, where V takes the value $\frac{6}{5}$ [see (2.14)]. This makes it plausible that $d = \frac{6}{5}$ and that $v_{2,b}$ provides the mountain pass out of the well. In order to prove rigorously that this is the case, we would have to show that $v_{2,b}$ is not only a critical point for (4.2) but also a minimum, something we have not attempted. Note that, be that as it may, it is certainly true that $d \leqslant \frac{6}{5}$.

V. EXISTENCE AND NONEXISTENCE OF SOLUTIONS

Fourier analysis reveals that the linearization of (2.2) given by

$$w_t = -u_{xxx} + u_x, \quad u_t = w_x,$$

generates a strongly continuous semigroup in the space $L^2 \times H^1$. Since the nonlinear mapping $(w,u) \to ((u^2)_x, 0)$ is indefinitely continuously differentiable in that product space, the results on nonlinear semigroups of Segal[8] show that (2.2) has a generalized solution $(w(t),u(t))$ for each initial data $(w(0),u(0))$ in $L^2 \times H^1$. This solution exists for a positive length of time [depending on $(w(0),u(0))$] and is a continuous function of t and $(w(0),u(0))$. Furthermore, $(w(t),u(t))$ is smooth if $w(0)$ and $u(0)$ are. The functionals M and E in (2.3)–(2.6) are conserved by generalized solutions, since these functionals are continuous in $L^2 \times H^1$ and conserved by smooth solutions. Therefore, if a generalized solution satisfies $E(0) < d$ and $u(0) \in W$, then $u(t)$ remains in the well throughout its interval of existence. Then (4.4) implies that $\|u\|_{H^1}$ remains bounded; hence the solution exists for all times $0 < t < \infty$ (Segal[8]). On the other hand, solutions for which $u(0)$ is not in W and solutions for which the energy E exceeds the well depth d are not likely to exist for all time. This is exemplified by the case where the initial condition consists of two solitons, well separated and moving towards each other. In Fig. 2 the dashed line represents the locus

$$\tfrac{6}{5}P_1^3(5-4P_1^2) + \tfrac{6}{5}P_2^3(5-4P_2^2) = \tfrac{6}{5} > d.$$

Thus, according to (2.12), $E > d$ to the right of the dashed line, a domain which includes region II, which was shown to correspond to $S + S \rightarrow A + A$ interactions and therefore exhibits blowup in finite time.

Sufficient conditions for the blowup to occur can be obtained by concavity arguments (Payne[9]). For instance, smooth solutions $(w(t),u(t))$ that vanish at $x = \pm\infty$, together with their derivative, cannot exist for all time if $E < 0$ and $I = 0$. Elimination of w in (2.2) reveals that such solutions satisfy

$$-D^{-2}u_{tt} = u_{xx} - u - u^2,$$

with

$$D^{-2} = \int_{-\infty}^{x}\int_{-\infty}^{x} \cdot$$

Since the operator $-D^{-2}$ acting on the indicated class of functions is symmetric and positive definite, the results in Payne[9], Sec. 8, establish that the existence time of the solution is necessarily finite.

VI. DISCUSSION

We have shown that the "good" Boussinesq equation possesses a highly complicated mechanism for solitary wave interaction. Three salient features are that solitary waves exist only for a finite range of velocities, that interactions can alter the nature of the solitary waves, and that merging and splitting are possible. These properties may, no doubt, be of interest in modeling. In this connection it would be useful to carry out a stability analysis of the solitons of (2.2) (cf. Benjamin[10]). However, solitons with a large value of P are certainly orbitally unstable. This can be seen in Fig. 2, where it is apparent that perturbing the soliton of parameter $P_1 = \sqrt{3}/2$ with any other soliton, no matter how small, always leads to an $S + S \rightarrow A + A$ interaction, and, therefore, to blowup in finite time. Also, solitons with $P_1 > \sqrt{3}/2$ can split, under arbitrarily small perturbations, into either two solitons or two antisolitons with parameters P_2 and $P_1 - P_2$, where (P_1, P_2) lies in the arc AB. On the other hand, numerical evidence has led us to conjecture that small-amplitude solitons are orbitally stable. This conjecture can be proved for the periodic problem for (2.2). The details will be reported elsewhere.

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