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Nonlinear stability and convergence of finite-difference methods for the "good" Boussinesq equation

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Summary. The "good" Boussinesq equation $u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx}$ has recently been found to possess an interesting soliton-interaction mechanism. In this paper we study the nonlinear stability and the convergence of some simple finite-difference schemes for the numerical solution of problems involving the "good" Boussinesq equation. Numerical experiments are also reported.

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1 Introduction

It has recently been discovered [7] that the interactions of solitary-wave solutions of the "good" Boussinesq equation

$$u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx}$$

obey a highly interesting mechanism. The analytic expression of such solutions is

$$(1.1) \qquad u(x,t) = -A\,\mathrm{sech}^2[(P/2)(\xi - \xi_0)], \qquad \xi = x - ct;$$

where ξ_0 and $P > 0$ are free real parameters and the amplitude A and velocity c of the wave are related to P through the formulas

$$(1.2) \qquad A = 3P^2/2, \qquad c = \pm\sqrt{1 - P^2}.$$

Note that ξ_0 determines the initial position of the wave, and that, due to the square root in (1.2), the parameter P can only take values in $0 < P \leqq 1$. Thus, the solitary waves (1.1) only exist for a finite range of velocities $-1 < c < 1$. Of course, a positive (respectively negative) velocity corresponds to a wave moving to the right (respectively to the left).

When two solitary waves with parameters P_1 and P_2 are initially well separated and approach each other, a nonlinear interaction takes place. If P_1 and P_2 are of moderate size, the incoming waves emerge of the interaction without

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changing shape or velocity. However, if P_1 and P_2 are large, the interaction leads to the creation of singular solutions, which have been studied in [7]. In the border-line case between the two situations previously described, the incoming waves merge into a single wave with parameter $P_3 = P_1 + P_2$.

To sum up, the "good" Boussinesq equation is similar to the Korteweg-de Vries (KdV) or cubic Schroedinger (CS) equations, in that it gives rise to solitons. However it differs from these well-known equations in a number of features: finite-range of velocities for the solitary waves, possibility of interactions leading to singular solutions and possibility of two waves merging into a single one.

While for the KdV and CS equations the available literature, both numerical and analytical, is very large, the study of the "good" Boussinesq equation is only beginning. Two recent papers, containing references to earlier work, are [6] and [7]. In [6] an exact formula is given for the interaction of solitary waves. Numerical experience is also reported. The article [7] studies in detail the interaction mechanism and discusses the existence and regularity of solutions. However further investigations are needed, particularly as far as the stability of the solutions is concerned [8]. Clearly such investigations should combine numerical studies with analytical techniques.

This paper is devoted to the analysis of finite-difference methods for the numerical integration of the "good" Boussinesq equation. For the sort of problems considered here (one-dimensional, smooth waves), finite-difference techniques are often (see e.g. [13, 14]) judged not to be competitive with spectral and pseudo-spectral methods. However, it is fair to say that the finite-difference schemes presented in this paper are very simple to code and may be easily modified to cater for a variety of boundary conditions. Therefore, such schemes may provide convenient numerical methods if high accuracy is not required.

The organization of the article is as follows. In Sect. 2 we present the problem to be solved. In the Sect. 3 we introduce a simple explicit scheme. Its nonlinear stability and convergence are proved in Sect. 4. Section 5 is devoted to the analysis of unconditionally stable implicit schemes. The final Section presents some numerical experiments.

2 The "good" Boussinesq equation

We consider the periodic problem

$$u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx}, \qquad -\infty < x < \infty, \quad 0 \leqq t \leqq T < \infty, \tag{2.1}$$

$$u(x,t) = u(x+1,t), \qquad -\infty < x < \infty, \quad 0 \leqq t \leqq T, \tag{2.2}$$

$$u(x,0) = u^0(x), \qquad -\infty < x < \infty, \tag{2.3}$$

$$u_t(x,0) = v^0(x), \qquad -\infty < x < \infty, \tag{2.4}$$

where the data u^0, v^0 are 1-periodic functions, which are assumed to be smooth enough for (2.1)–(2.4) to have a unique solution (cf. [7, 8]).

The quadratic functional

$$\|u_t\|^2 + \|u_{xx}\|^2, \tag{2.5}$$

where $\|\cdot\|$ denotes the standard L^2-norm for 1-periodic functions, is an invariant of motion of the problem given by (2.2)–(2.4) along with

$$u_{tt} = -u_{xxxx}, \tag{2.6}$$

i.e. of the periodic initial-value problem for the linear principal part of the "good" Boussinesq equation (2.1). The conservation of (2.5) is proved by multiplication by u_t in (2.6) and integration by parts.

Another conservation law for (2.2)–(2.4), (2.6) is given by

$$\int_0^1 u_t(x,t)\,\mathrm{d}x = \int_0^1 v^0(x)\,\mathrm{d}x,$$

leading to

$$\int_0^1 u(x,t)\,\mathrm{d}x = t\int_0^1 v^0(x)\,\mathrm{d}x + \int_0^1 u^0(x)\,\mathrm{d}x. \tag{2.7}$$

Combining the conservations of (2.5), (2.7), using the Cauchy-Schwartz inequality and denoting $u_t = v$, we arrive at

$$\begin{aligned}(\|v(.,t)\|^2 + \|u_{xx}(.,t)\|^2)^{1/2} + \left|\int_0^1 u(x,t)\,\mathrm{d}x\right| \\ \leqq (1+t)\left\{(\|v^0\|^2 + \|u_{xx}^0\|^2)^{1/2} + \left|\int_0^1 u^0(x)\,\mathrm{d}x\right|\right\};\end{aligned} \tag{2.8}$$

so that the 1-periodic, initial-value problem for (2.6) is well posed in the following energy norm for pairs (v, u) of 1-periodic functions belonging to $L_p^2 \times H_p^2$,

$$\|(v,u)\|_E = (\|v\|^2 + \|u_{xx}\|^2)^{1/2} + \left|\int_0^1 u\,\mathrm{d}x\right|. \tag{2.9}$$

Now the well-known inequalities

$$\|w\|^2 \leqq \frac{1}{4\pi^2}\|w_x\|^2 + \left(\int_0^1 w\,\mathrm{d}x\right)^2, \tag{2.10}$$

$$\|w_x\|^2 \leqq \frac{1}{4\pi^2}\|w_{xx}\|^2, \tag{2.11}$$

valid for $w \in H_p^2$, clearly imply that the norm in (2.9) is equivalent to the Sobolev norm

$$(\|v\|^2 + \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2)^{1/2}. \tag{2.12}$$

Therefore (2.8) shows that (2.2)–(2.4), (2.6) is well posed $L_p^2 \times H_p^2$. The nonlinear finite-difference analyses to be presented later are based on the use of discrete analogues of the energy norm in (2.9).

3 An explicit scheme and its energy norm

Let J be a positive integer and set $h=1/J$. We denote by $\mathbb{Z}_h$ the space of real, 1-periodic functions defined on the grid $\{x_j\colon x_j=jh,\ j=0, \pm 1, \pm 2, \ldots\}$. Thus each element $\mathbf{V}\in\mathbb{Z}_h$ is a sequence $\{V_j\}_{j=0,\pm 1,\ldots}$ with $V_j=V_{j+J}, j=0, \pm 1, \ldots$ We need the following finite-difference operators with domain and range equal to $\mathbb{Z}_h$.

(3.1) $$(T_+\mathbf{V})_j=V_{j+1}, \qquad j=0, \pm 1, \pm 2, \ldots$$

(3.2) $$(T_-\mathbf{V})_j=V_{j-1}, \qquad j=0, \pm 1, \pm 2, \ldots$$

(3.3) $$(D_+\mathbf{V})_j=(V_{j+1}-V_j)/h, \qquad j=0, \pm 1, \pm 2, \ldots$$

(3.4) $$(D_-\mathbf{V})_j=(V_j-V_{j-1})/h, \qquad j=0, \pm 1, \pm 2, \ldots$$

(3.5) $$D^2=D_+D_-,$$

(3.6) $$D^4=D^2D^2.$$

Next, let k denote a parameter $0<k<T$ and consider the time-levels $t_n=nk$, $n=0, 1, \ldots, N$, with $N=[T/k]$. In the sequel a superscript n denotes a quantity associated with the time-level t_n. With these notations, we consider the finite-difference scheme

(3.7) $$(\mathbf{U}^{n+1}-2\mathbf{U}^n+\mathbf{U}^{n-1})/k^2=-D^4\mathbf{U}^n+D^2\mathbf{U}^n+D^2(\mathbf{U}^n)^2, \qquad n=0, 1, \ldots, N-1$$

with the periodicity condition

(3.8) $$\mathbf{U}^n\in\mathbb{Z}_h, \qquad n=0, 1, \ldots, N-1$$

and initial values

(3.9) $$\mathbf{U}^0=\boldsymbol{\alpha},$$

(3.10) $$(\mathbf{U}^1-\mathbf{U}^0)/k=\boldsymbol{\beta},$$

where $\boldsymbol{\alpha}\in\mathbb{Z}_h$, $\boldsymbol{\beta}\in\mathbb{Z}_h$ are given approximations to the grid-restrictions of the functions u^0 and v^0 in (2.3), (2.4).

The remainder of this section deals with the construction of a discrete analogue of the energy norm (2.9). We first introduce a discrete analogue Q_k of the quadratic functional (2.5). If $(\mathbf{W}, \mathbf{W}^*)\in\mathbb{Z}_h\times\mathbb{Z}_h$ we set

(3.11) $$Q_k(\mathbf{W}, \mathbf{W}^*)=\|(\mathbf{W}-\mathbf{W}^*)/k\|^2+(D^2\mathbf{W}, D^2\mathbf{W}^*),$$

where $\|\cdot\|$ denotes the standard L^2-norm

$$\|\mathbf{W}\|^2=\Sigma_{1\leq j\leq J}\, h\, W_j^2$$

and $(.,.)$ represents the corresponding inner product. (Note that the same symbol $\|\cdot\|$ is used for the continuous and discrete cases, but no confusion is possible.)

The definition of the quadratic form Q_k in (3.11) is motivated by the fact that, if $\mathbf{U}^n \in \mathbb{Z}_h$, $n=0, 1, \ldots, N$ satisfy

$$(3.12) \qquad (\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1})/k^2 = -D^4 \mathbf{U}^n, \qquad n=0, 1, \ldots, N-1,$$

(principal part of (3.7)), then

$$Q_k(\mathbf{U}^{n+1}, \mathbf{U}^n) = Q_k(\mathbf{U}^1, \mathbf{U}^0), \qquad n=0, 1, \ldots, N-1.$$

This follows easily by induction, after taking the inner product of (3.12) and $(\mathbf{U}^{n+1} - \mathbf{U}^n) + (\mathbf{U}^n - \mathbf{U}^{n-1})$ and noticing that in (3.6) the operator D^2 is selfadjoint.

After introducing the averaging functional I

$$I(\mathbf{W}) = \Sigma_{1 \leqq j \leqq J} h W_j, \qquad \mathbf{W} \in \mathbb{Z}_h,$$

we define the energy

$$(3.13) \qquad \|(\mathbf{W}, \mathbf{W}^*)\|_E = Q_k(\mathbf{W}, \mathbf{W}^*)^{1/2} + |I(\mathbf{W})|, \qquad (\mathbf{W}, \mathbf{W}^*) \in \mathbb{Z}_h \times \mathbb{Z}_h,$$

for which it is easily shown that the solutions of (3.12) possess an estimate similar to (2.8). However such an estimate will not be presented here because it plays no role in the analysis of (3.7) to be carried out in Sect. 4. We rather turn to the study of the properties of (3.13).

Proposition 3.1. Assume that $r := k/(h^2) < 1/2$. Then there exists a positive constant C_1, depending only on r, such that for $(\mathbf{W}, \mathbf{W}^*) \in \mathbb{Z}_h \times \mathbb{Z}_h$

$$(3.14) \qquad \begin{aligned} Q_k(\mathbf{W}, \mathbf{W}^*) &\leqq \|(\mathbf{W} - \mathbf{W}^*)/k\|^2 + (1/2)\|D^2 \mathbf{W}\|^2 + (1/2)\|D^2 \mathbf{W}^*\|^2 \\ &\leqq C_1 Q_k(\mathbf{W}, \mathbf{W}^*). \end{aligned}$$

Proof. The first inequality in (3.14) is obvious. To prove the second, we introduce the following quadratic form P_k in $\mathbb{Z}_h \times \mathbb{Z}_h$

$$P_k(\mathbf{W}, \mathbf{W}^*) = \|(\mathbf{W} - \mathbf{W}^*)/k\|^2 + (1/2)\|D^2 \mathbf{W}\|^2 + (1/2)\|D^2 \mathbf{W}^*\|^2$$

and compare the eigenvalues/vectors of P_k and Q_k.

From (3.11), the self-adjoint operator in $\mathbb{Z}_h \times \mathbb{Z}_h$ associated with Q_h may be writen in block form as

$$\begin{bmatrix} k^{-2} I & -k^{-2} I + \frac{1}{2} D^4 \\ -k^{-2} I + \frac{1}{2} D^4 & k^{-2} I \end{bmatrix}$$

therefore, if $(\mathbf{W}, \mathbf{W}^*)$ is an eigenfunction associated with the eigenvalue λ, then

$$\begin{aligned} k^{-2}\mathbf{W} - k^{-2}\mathbf{W}^* + (1/2) D^4 \mathbf{W}^* &= \lambda \mathbf{W}, \\ -k^{-2}\mathbf{W} + (1/2) D^4 \mathbf{W} + k^{-2}\mathbf{W}^* &= \lambda \mathbf{W}^*. \end{aligned}$$

By adding and subtracting these equations we obtain

$$(3.15) \qquad (1/2) D^4 (\mathbf{W} + \mathbf{W}^*) = \lambda (\mathbf{W} + \mathbf{W}^*),$$

$$(3.16) \qquad 2k^{-2}(\mathbf{W} - \mathbf{W}^*) - (1/2) D^4 (\mathbf{W} - \mathbf{W}^*) = \lambda (\mathbf{W} - \mathbf{W}^*).$$

When eigenfunctions with $\mathbf{W}=\mathbf{W}^*$ are looked for, (3.16) holds and (3.15) implies that $\mathbf{W}$ is an eigenfunction of $(1/2)D^4$ with eigenvalue λ. This provides J eigenvalues of Q_k. On the other hand, if eigenfunctions with $\mathbf{W}=-\mathbf{W}^*$ are sought, (3.15) holds and (3.16) reveals that λ is of the form $2k^{-2}-\mu$ with μ an eigenvalue of $(1/2)D^4$ and $\mathbf{W}$ the associated eigenfunction.

Turning now to the form P_k, a similar argument yields that the eigenvalues/functions of P_k are $\{\mu, (\mathbf{W}+\mathbf{W}^*)\}$, $\{2k^{-2}+\mu, (\mathbf{W}-\mathbf{W}^*)\}$ with $\{\mu, \mathbf{W}\}$ the eigenvalues/functions of $(1/2)D^4$. Since P_k, Q_k possess a common set of eigenfunctions, the second inequality in (3.14) is equivalent to the condition

$$\lambda(P_k)\leqq C_1\,\lambda(Q_k)$$

for the corresponding eigenvalues, or

$$2k^{-2}+\mu, \leqq C_1(2k^{-2}-\mu), \qquad \mu\in\operatorname{Spec}(1/2)D^4. \tag{3.17}$$

Fourier analysis shows that the eigenvalues of the operator D^2 in (3.5) are $2h^{-2}(\cos 2\pi jh-1)$, $j=1,2,\ldots,J$, so that, according to (3.6), the eigenvalues of $1/2D^4$ are $2h^{-4}(\cos 2\pi jh-1)^2$, $j=1,2,\ldots,J$. Therefore $\operatorname{Spec}\{(1/2)D^4\}\subset[0, 8h^{-4}]$ and (3.17) holds with $C_1=(1+4r^2)/(1-4r^2)$, provided that $r^2<1/4$, i.e. $r<1/2$.

Corollary. If $r=k/(h^2)<1/2$, then the expression in (3.13) defines a norm in $\mathbb{Z}_h\times\mathbb{Z}_h$.

Proof. The proposition shows that $\|\cdot\|_E$ is a seminorm. If $\|(\mathbf{W},\mathbf{W}^*)\|_E=0$, then (3.14) implies that $D^2\mathbf{W}=0$, which, taking into account the periodicity of $\mathbf{W}$, shows that $\mathbf{W}$ must be a constant grid-function. On the other hand $I(\mathbf{W})=0$, so that, in fact $\mathbf{W}=0$. Finally (3.14) reveals that $\mathbf{W}^*=\mathbf{W}=\mathbf{0}$.

In [8] a proof is given of the following lemma which provides counterparts of the inequalities (2.10)–(2.11):

Lemma. *For* $\mathbf{W}\in\mathbb{Z}_h$

$$\|\mathbf{W}\|^2\leqq(1/4)\|D_-\mathbf{W}\|^2+I(\mathbf{V})^2, \tag{3.18}$$

$$\|D_-\mathbf{W}\|^2\leqq(1/4)\|D^2\mathbf{W}\|. \tag{3.19}$$

The inequalites (3.18), (3.19), (3.14) readily imply that, for $r<1/2$, the energy norm (3.13) is equivalent to a discrete Sobolev norm (cf. (2.12)). Namely:

Proposition 3.2. *Assume that* $r=k/(h^2)<1/2$. *Then there exist a positive constant* C_2, *depending only on* r, *and a positive constant* C_3 *(independent of* k, h *and* r*) such that for* $(\mathbf{W},\mathbf{W}^*)\in\mathbb{Z}_h\times\mathbb{Z}_h$

$$\begin{aligned} C_3^2\,\|(\mathbf{W},\mathbf{W}^*)\|_E^2 &\leqq \|(\mathbf{W}-\mathbf{W}^*)/k\|^2+\|\mathbf{W}\|^2+\|D_-\mathbf{W}\|^2+\|D^2\mathbf{W}\|^2 \\ &\quad+\|D^2\mathbf{W}^*\|^2\leqq C_2^2\,\|(\mathbf{W},\mathbf{W}^*)\|_E^2. \end{aligned} \tag{3.20}$$

Finally we shall employ the Sobolev-imbedding inequalities [8]

$$\|\mathbf{W}\|_\infty^2 \leqq (1/2)\|D^2\mathbf{W}\|^2 + (5/2)\|\mathbf{W}\|^2, \qquad \mathbf{W} \in \mathbb{Z}_h, \tag{3.21}$$

$$\|D_-\mathbf{W}\|_\infty^2 \leqq 2\|D^2\mathbf{W}\|^2 + \|\mathbf{W}\|^2, \qquad \mathbf{W} \in \mathbb{Z}_h, \tag{3.22}$$

together with the inverse estimate

$$\|\mathbf{W}\|_\infty \leqq h^{-1/2}\|\mathbf{W}\|, \qquad \mathbf{W} \in \mathbb{Z}_h. \tag{3.23}$$

4 Nonlinear stability and convergence

To investigate the stability and convergence of the scheme (3.7)–(3.10) we employ a general analytical framework introduced by López-Marcos and Sanz-Serna [9, 3–5]. The cornerstone of this framework is an important lemma due to Stetter [11, Lemma 1.2.1], whose use avoids the need for establishing a priori bounds in convergence proofs of nonlinear algorithms [1]. Furthermore, when using this general formalism on implicit discretizations, there is no need to provide a proof of the existence of discrete solutions separate from the proof of convergence.

To facilitate the readibility of the subsequent analysis, we first present a very brief summary of the general definitions and main results of [3–5]. This is followed by a study of the stability, consistency and convergence of (3.7)–(3.10). Other instances of use of the general concepts employed here can be seen in e.g. [2, 12].

Discretization framework

Consider a fixed, given problem concerning a differential or integral equation. Let u be a solution of this problem. We denote by $\mathbf{U}_h$ the numerical approximation to u. The subscript h shows that $\mathbf{U}_h$ depends on a small parameter h, such as a mesh-size. We assume that h takes values in a set H of positive numbers with $\inf H = 0$. The numerical approximation $\mathbf{U}_h$ is obtained, for each fixed h in H, by solving a *discrete problem*.

$$\Phi_h(\mathbf{U}_h) = \mathbf{0}, \tag{4.1}$$

where Φ_h is a mapping with domain $D_h \subset X_h$ and values in Y_h. Here X_h and Y_h are normed spaces, both real or both complex, with the same finite dimension.

To investigate how close $\mathbf{U}_h$ is to u, we choose, for each h in H, an element $\mathbf{u}_h$ in D_h. This element is a suitable discrete representation of u. Typically, in a difference method, $\mathbf{u}_h$ will be a set of nodal values of u. The *global discretization error* is defined to be the vector $\mathbf{e}_h = \mathbf{u}_h - \mathbf{U}_h$ and the *local discretization error* is given by $\mathbf{l}_h = \Phi_h(\mathbf{u}_h)$. We say that the discretization (4.1) is convergent if there exists $h_0 > 0$, such that for h in H, $h \leqq h_0$, (4.1) has a solution $\mathbf{U}_h$ in such a way that, as $h \to 0$, $\lim \|\mathbf{u}_h - \mathbf{U}_h\| = 0$. The convergence is of order p, if $\|\mathbf{u}_h - \mathbf{U}_h\| = \mathcal{O}(h^p)$. The discretization (4.1) is consistent (respectively consistent of order p) if, as $h \to 0$, $\lim \|\Phi_h(\mathbf{u}_h)\| = o(1)$ (respectively $\mathcal{O}(h^p)$).

Assume that for each h in H, R_h is a value with $0<R_h\leqq+\infty$. We say that (4.1) is stable *restricted to the thresholds* R_h, if there exist two positive constants h_0 and S such that for any h in H, $h\leqq h_0$, the open ball $B(\mathbf{u}_h, R_h)$ is contained in the domain D_h and for any $\mathbf{V}_h$, $\mathbf{W}_h$ in this ball

$$\|\mathbf{V}_h-\mathbf{W}_h\|\leqq S\,\|\Phi_h(\mathbf{V}_h)-\Phi_h(\mathbf{W}_h)\|. \tag{4.2}$$

It should be emphasized that the stability bound (4.2) has to be proved not for arbitrary $\mathbf{V}_h$ and $\mathbf{W}_h$, but only for vectors $\mathbf{V}_h$ and $\mathbf{W}_h$ "near" the theoretical solution; near in the sense that $\|\mathbf{V}_h-\mathbf{u}_h\|<R_h$, $\|\mathbf{W}_h-\mathbf{u}_h\|<R_h$. Thus, this notion of stability is weaker than other available [3–5]. However stability and consistency still imply convergence. Namely:

Theorem 4.1. *Assume that* (4.1) *is consistent and stable with thresholds* R_h. *If* Φ_h *is continuous in* $B(\mathbf{u}_h, R_h)$ *and* $\|\mathbf{l}_h\|=o(R_h)$ *as* $h\to 0$, *then:*

(i) *For h small enough, the discrete* (4.1) *possess a unique solution in* $B(\mathbf{u}_h, R_h)$.

(ii) *As* $h\to 0$ *the solutions in* (i) *converge with an order of convergence not smaller than the order of consistency.*

We write the scheme (3.7)–(3.10) within the previous abstract framework as follows.

(i) First of all, only one discretization parameter is allowed in the abstract framework, so that a relation between k and h should be imposed. We assume that $k=rh^2$ with r a fixed constant.

(ii) We take $X_h=Y_h=\mathbb{Z}_h^{N+1}$. In X_h we use a maximum norm

$$\|\mathbf{W}_h\|_{X_h}=\max\{\|(\mathbf{W}^{n+1},\mathbf{W}^n)\|_E: 0\leqq n\leqq N-1\},\qquad \mathbf{W}_h=[\mathbf{W}^0,\mathbf{W}^1,\dots,\mathbf{W}^N]\in X_h$$

and in Y_h we employ an L^1-norm

$$\|\mathbf{G}_h\|_{Y_h}=\|(\mathbf{G}^1,\mathbf{G}^0)\|_E+\sum_{n=2}^{N} k\,\|\mathbf{G}^n\|,\qquad \mathbf{G}_h=[\mathbf{G}^0,\mathbf{G}^1,\dots\mathbf{G}^N]\in Y_h.$$

For the relevance of using L^∞, L^1 norms in initial value problems and the relation with the familiar Lax stability see [9, 10].

(iii) On defining the mapping Φ_h given by $\Phi_h(\mathbf{W}_h)=\mathbf{G}_h$ with

$$\mathbf{G}^{n+1}=k^{-2}(\mathbf{W}^{n+1}-2\,\mathbf{W}^n+\mathbf{W}^{n-1})+D^4\mathbf{W}^n-D^2\mathbf{W}^n-D^2(\mathbf{W}^n)^2,\qquad 1\leqq n\leqq N-1,$$

$$\mathbf{G}^1=\mathbf{W}^1-\boldsymbol{\alpha}-k\boldsymbol{\beta},$$

$$\mathbf{G}^0=\mathbf{W}^0-\boldsymbol{\alpha},$$

the problem (3.7)–(3.10) adopts the abstract form (3.1). Each of the $N+1$ components of Φ_h corresponds to the computation of a time level.

(iv) Finally the discrete representation of the theoretical solution u is given by the obvious choice

$$\mathbf{u}_h=[\mathbf{r}_h u^0, \mathbf{r}_h u^1, \ldots, \mathbf{r}_h u^N],$$

where u^n is the function $u(., t_n)$, $n=0, 1, \ldots, N$ and $\mathbf{r}_h$ denotes the grid restriction operator (with values in $\mathbb{Z}_h$).

Consistency

Simple Taylor expansions yield:

Theorem 4.2. *Assume that the solution u of* (2.1)–(2.4) *possesses bounded derivatives* $\partial^4 u/\partial t^4$, $\partial^6 u/\partial x^6$, $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$. *Then the local error of the discretization* (3.7)–(3.10) *satisfies*

$$\|\Phi_h(\mathbf{u}_h)\|_{Y_h} \leqq \|(\mathbf{U}^1-\mathbf{r}_h u^1, \mathbf{U}^0-\mathbf{r}_h u^0)\|_E + C(k^2+h^2), \tag{4.3}$$

where C is a constant depending only on u and T.

Note that the choice

$$\boldsymbol{\alpha}=\mathbf{r}_h u^0, \qquad \boldsymbol{\beta}=\mathbf{r}_h v^0+(k/2)\,\mathbf{r}_h u_{tt}(., 0) \tag{4.4}$$

in (3.9)–(3.10) leads to

$$\|(\mathbf{U}^1-\mathbf{r}_h u^1, \mathbf{U}^0-\mathbf{r}_h u^0)\|_E=\mathcal{O}(k^2+h^2), \qquad h \to 0, \tag{4.5}$$

and therefore to consistency of the second order. Of course u_{tt} in (4.4) is available from u^0, v^0 and the differential equation.

Stability

The key result of this section is the following:

Theorem 4.3. *Assume that* $r=kh^{-2}<1/2$ *and that the derivatives* u_x, u_{xx} *of the solution u of* (2.1) *are bounded for* $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$. *Fix a constant* $\mu>0$. *Then the discretization* (3.7)–(3.10) *is stable with thresholds* $R_h=\mu h^{1/2}$.

Proof. Let $\mathbf{V}_h$, $\mathbf{W}_h$ in $B(\mathbf{u}_h, \mu h^{1/2})$, $\mathbf{V}_h=[\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N]$, $\mathbf{W}_h=[\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N]$ and set $\Phi_h(\mathbf{V}_h)=[\mathbf{F}^0, \mathbf{F}^1, \ldots, \mathbf{F}^N]$, $\Phi_h(\mathbf{W}_h)=[\mathbf{G}^0, \mathbf{G}^1, \ldots, \mathbf{G}^N]$. Then, by definition of Φ_h,

$$(\mathbf{V}^{n+1}-2\mathbf{V}^n+\mathbf{V}^{n-1})/k^2+D^4\mathbf{V}^n-D^2\mathbf{V}^n-D^2(\mathbf{V}^n)^2=\mathbf{F}^{n+1}, \qquad n=1, 2, \ldots, N-1,$$

$$(\mathbf{W}^{n+1}-2\mathbf{W}^n+\mathbf{W}^{n-1})/k^2+D^4\mathbf{W}^n-D^2\mathbf{W}^n-D^2(\mathbf{W}^n)^2=\mathbf{G}^{n+1},$$
$$n=1, 2, \ldots, N-1.$$

Subtract and rearrange to arrive at

$$(\mathbf{e}^{n+1}-2\mathbf{e}^n+\mathbf{e}^{n-1})/k^2+D^4\mathbf{e}^n=D^2\mathbf{e}^n+D^2[(\mathbf{V}^n)^2-(\mathbf{W}^n)^2]+\mathbf{L}^{n+1}, \tag{4.6}$$
$$n=1, 2, \ldots, N-1,$$

where we have used the notation $\mathbf{e}^n = \mathbf{V}^n - \mathbf{W}^n$, $\mathbf{L}^n = \mathbf{F}^n - \mathbf{G}^n$, $n = 0, 1, \ldots, N$. Next, take the inner product of (4.6) and $(\mathbf{e}^{n+1} - \mathbf{e}^n) + (\mathbf{e}^n - \mathbf{e}^{n-1})$ to arrive at (cf. (3.11)):

$$\begin{aligned} &Q_k(\mathbf{e}^{n+1} - \mathbf{e}^n) - Q_k(\mathbf{e}^n - \mathbf{e}^{n-1}) \\ &\quad = (D^2 \mathbf{e}^n + D^2 [(\mathbf{V}^n)^2 - (\mathbf{W}^n)^2] + \mathbf{L}^{n+1}, (\mathbf{e}^{n+1} - \mathbf{e}^n) + (\mathbf{e}^n - \mathbf{e}^{n-1})) \\ &\quad \leqq k \, \| D^2 \mathbf{e}^n + D^2 [(\mathbf{V}^n)^2 - (\mathbf{W}^n)^2] + \mathbf{L}^{n+1} \| \, \{ \| (\mathbf{e}^{n+1} - \mathbf{e}^n)/k \| + \| (\mathbf{e}^n - \mathbf{e}^{n-1})/k \| \}. \end{aligned}$$

The definition of Q_k in (3.11) implies

$$Q_k(\mathbf{e}^{n+1} - \mathbf{e}^n) \geqq \| (\mathbf{e}^{n+1} - \mathbf{e}^n)/k \|, \qquad n = 0, 1, \ldots, N,$$

and therefore

$$\begin{aligned} &Q_k(\mathbf{e}^{n+1} - \mathbf{e}^n)^{1/2} - Q_k(\mathbf{e}^n - \mathbf{e}^{n-1})^{1/2} \\ &\quad \leqq k \, \| D^2 \mathbf{e}^n \| + \| D^2 [(\mathbf{V}^n)^2 - (\mathbf{W}^n)^2] \| + \| \mathbf{L}^{n+1} \|. \end{aligned} \tag{4.7}$$

In the remainder of the proof K denotes a constant independent of k, h (K may depend on u, T, μ and r and may have different values at different occurrences).

To bound the right hand side of (4.7) we first recall that the Proposition 3.2 implies

$$\| D^2 \mathbf{e}^n \| \leqq \| (\mathbf{e}^n, \mathbf{e}^{n-1}) \|_E.$$

On the other hand, with the notations in (3.1)–(3.5)

$$\begin{aligned} D^2 [(\mathbf{V}^n)^2 - (\mathbf{W}^n)^2] &= D^2 [(\mathbf{V}^n + \mathbf{W}^n) \mathbf{e}^n] \\ &= T_+ (\mathbf{V}^n + \mathbf{W}^n) D^2 \mathbf{e}^n + 2 D_+ (\mathbf{V}^n + \mathbf{W}^n) D_- \mathbf{e}^n \\ &\quad + D^2 (\mathbf{V}^n + \mathbf{W}^n) T_- \mathbf{e}^n, \end{aligned}$$

so that

$$\begin{aligned} \| D^2 [(\mathbf{V}^n)^2 - (\mathbf{W}^n)^2] \| &\leqq \| (\mathbf{V}^n + \mathbf{W}^n) \|_\infty \| D^2 \mathbf{e}^n \| + 2 \| D_+ (\mathbf{V}^n + \mathbf{W}^n) \|_\infty \| D_- \mathbf{e}^n \| \\ &\quad + \| D^2 (\mathbf{V}^n + \mathbf{W}^n) \|_\infty \| \mathbf{e}^n \|. \end{aligned} \tag{4.8}$$

The second bound in (3.20) leads to

$$\begin{aligned} \| D^2 \mathbf{e}^n \| &\leqq \| (\mathbf{e}^n, \mathbf{e}^{n-1}) \|_E \\ \| D_- \mathbf{e}^n \| &\leqq \| (\mathbf{e}^n, \mathbf{e}^{n-1}) \|_E, \\ \| \mathbf{e}^n \| &\leqq \| (\mathbf{e}^n, \mathbf{e}^{n-1}) \|_E, \end{aligned}$$

while by (3.20)–(3.23) and the threshold condition

$$\| (\mathbf{V}^n + \mathbf{W}^n) \|_\infty \leqq 2 \| \mathbf{r}_h u^n \|_\infty + \| \mathbf{V}^n - \mathbf{r}_h u^n \|_\infty + \| \mathbf{W}^n - \mathbf{r}_h u^n \|_\infty \leqq K,$$

$$\| D_+ (\mathbf{V}^n + \mathbf{W}^n) \|_\infty \leqq 2 \| D_+ \mathbf{r}_h u^n \|_\infty + \| D_+ (\mathbf{V}^n - \mathbf{r}_h u^n) \|_\infty + \| D_+ (\mathbf{W}^n - \mathbf{r}_h u^n) \|_\infty \leqq K,$$

$$\begin{aligned} \| D^2 (\mathbf{V}^n + \mathbf{W}^n) \|_\infty &\leqq 2 \| D^2 \mathbf{r}_h u^n \|_\infty + \| D^2 (\mathbf{V}^n - \mathbf{r}_h u^n) \|_\infty + \| D^2 (\mathbf{W}^n - \mathbf{r}_h u^n) \|_\infty \\ &\leqq K + h^{-1/2} \| D^2 (\mathbf{V}^n - \mathbf{r}_h u^n) \| + h^{-1/2} \| D^2 (\mathbf{W}^n - \mathbf{r}_h u^n) \| \\ &\leqq K + h^{-1/2} K \mu h^{1/2} = K. \end{aligned}$$

Returning now to (4.7)

$$Q_k(\mathbf{e}^{n+1}, \mathbf{e}^n)^{1/2} \leqq Q_k(\mathbf{e}^n, \mathbf{e}^{n-1})^{1/2} + kK \,\|(\mathbf{e}^n, \mathbf{e}^{n-1})\|_E + k\,\|\mathbf{L}^n\|. \tag{4.8}$$

Summation in (4.6) and rearrangement leads to

$$I(\mathbf{e}^{n+1}) = I(\mathbf{e}^n) + kI((\mathbf{e}^n - \mathbf{e}^{n-1})/k) + k^2 I(\mathbf{L}^{n+1})$$

and therefore

$$|I(\mathbf{e}^{n+1})| \leqq |I(\mathbf{e}^n)| + k\,\|(\mathbf{e}^n, \mathbf{e}^{n-1})\|_E + k^2\,\|\mathbf{L}^{n+1}\|$$

a bound which combined with (4.7) yields

$$\|(\mathbf{e}^{n+1}, \mathbf{e}^n)\|_E \leqq (1 + kK)\|(\mathbf{e}^n, \mathbf{e}^{n-1})\|_E + kK\,\|\mathbf{L}^{n+1}\|$$

and now a standard recursion concludes the proof.

Convergence

Theorem 4.4. *Assume that the solution u of* (2.1)–(2.4) *satisfies the smoothness assumptions of Theorem* 4.2 *and that the discrete initial data satisfy* (4.5). *Then if $r < 1/2$*

$$\max\{\|(\mathbf{U}^{n+1} - \mathbf{r}_h u^{n+1}, \mathbf{U}^n - \mathbf{r}_h u^n)\|_E, n = 0, 1, \ldots, N-1\} = \mathcal{O}(h^2), \qquad h \to 0.$$

In particular the estimates

$$\begin{aligned} \max\{\|\mathbf{U}^n - \mathbf{r}_h u^n\|_\infty + \|D_-(\mathbf{U}^n - \mathbf{r}_h u^n)\|_\infty \\ + \|D^2(\mathbf{U}^n - \mathbf{r}_h u^n)\|, n = 0, 1, \ldots, N-1\} = \mathcal{O}(h^2), \qquad h \to 0, \end{aligned} \tag{4.9}$$

are true, together with the following bound for the divided differences in time

$$\begin{aligned} \max\{\|\mathbf{r}_h[(u^{n+1} - u^n)/k] - [(\mathbf{U}^{n+1} - \mathbf{U}^n)/k], n = 0, 1, \ldots, N-1\} = \mathcal{O}(h^2), \\ h \to 0 \end{aligned} \tag{4.10}$$

Proof. The energy-norm estimate is a direct consequence of the Theorem 4.1. The Sobolev estimates then follow from (3.20)–(3.22).

A standard von Neumann analysis shows that for $r > 1/2$ the linearisation of the scheme (3.7) possesses normal modes with arbitrarily fast exponential growth. Therefore the restriction $r < 1/2$ in the theorems of this section is tight.

5 Implicit schemes

The explicit scheme (3.7) is conditionally stable. In this section we analyse a family of implicit schemes whose convergence does not require a condition $k = \mathcal{O}(h^2)$.

We consider the one-parameter family of methods

$$\begin{aligned} (\mathbf{U}^{n+1} - 2\,\mathbf{U}^n + \mathbf{U}^{n-1})/k^2 = \theta M\,\mathbf{U}^{n+1} + (1 - 2\,\theta) M\,\mathbf{U}^n + \theta M\,\mathbf{U}^{n-1}, \\ n = 1, 2, \ldots, N-1, \end{aligned} \tag{5.1}$$

where M represents the following finite difference operator in $\mathbb{Z}_h$

$$M\mathbf{V} = -D^4\mathbf{V} + D^2\mathbf{V} + D^2(\mathbf{V})^2, \qquad \mathbf{V}\in\mathbb{Z}_h,$$

and θ is a real parameter. The choice $\theta=0$ recovers the scheme of the previous sections.

The analysis of (5.1), (3.8)–(3.10) may be carried out following closely the techniques used in the study of the explicit scheme. Again a key step is the construction of a suitable energy norm. We now choose, instead of (3.13)

$$(5.2)\qquad \|(\mathbf{W},\mathbf{W}^*)\|_E^* = Q_k^*(\mathbf{W},\mathbf{W}^*)^{1/2} + |I(\mathbf{W})| + |I(\mathbf{W}^*)|, \qquad (\mathbf{W},\mathbf{W}^*)\in\mathbb{Z}_h\times\mathbb{Z}_h,$$

with Q_k^* defined

$$Q_k^*(\mathbf{W},\mathbf{W}^*) = \|(\mathbf{W}-\mathbf{W}^*)/k\|^2 + \theta\|D^2\mathbf{W}\|^2 + (1-2\theta)(D^2\mathbf{W}, D^2\mathbf{W}^*) + \theta\|D^2\mathbf{W}^*\|.$$

Again the choice of quadratic form is motivated by the fact that for solutions of

$$(\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1})/k^2 = \theta M\mathbf{U}^{n+1} + (1-2\theta)M\mathbf{U}^n + \theta M\mathbf{U}^{n-1}$$

the relations

$$Q_k^*(\mathbf{U}^{n+1},\mathbf{U}^n) = Q_k^*(\mathbf{U}^1,\mathbf{U}^0), \qquad n=0,1,\dots,N-1$$

hold.

For $\theta>1/4$, the role of Proposition 3.1 is now played by the following Cauchy-Schwartz estimations:

$$(5.3)\qquad \begin{aligned} K_1 Q_k^*(\mathbf{W},\mathbf{W}^*) &\leqq \|(\mathbf{W}-\mathbf{W}^*)/k\|^2 + (1/2)\|D^2\mathbf{W}\|^2 + (1/2)\|D^2\mathbf{W}^*\|^2 \\ &\leqq K_2 Q_k^*(\mathbf{W},\mathbf{W}^*), \qquad (\mathbf{W},\mathbf{W}^*)\in\mathbb{Z}_h\times\mathbb{Z}_h, \end{aligned}$$

with $K_1=1$, $K_2=1/(4\theta-1)$ if $1/2\geqq\theta>1/4$ and $K_1=1/(4\theta-1)$, $K_2=1$ if $\theta\geqq 1/2$. Thus, the Lemma in Sect. 3 implies that, for $\theta>1/4$, the energy norm (5.2) is equivalent to the following Sobolev norm

$$(5.4)\qquad \begin{aligned} (\|(\mathbf{W}-\mathbf{W}^*)/k\|^2 &+ \|\mathbf{W}\|^2 + \|D_-\mathbf{W}\|^2 \\ &+ \|D^2\mathbf{W}\|^2 + \|\mathbf{W}^*\|^2 + \|D_-\mathbf{W}^*\|^2 + \|D^2\mathbf{W}^*\|^2)^{1/2}. \end{aligned}$$

Note that this equivalence is uniform in k *and* h, while for the explicit scheme the equivalence between the energy and Sobolev norms was only uniform in h, with k restricted by $k/h^2=\text{constant}<1/2$. As a consequence, for $\theta>1/4$, the stability analysis of (5.1) may be performed assuming that, in the grid refinement, k varies as $k=\sigma(h)$, with σ an arbitrary increasing function such that $\sigma(0)=0$. Arguments very similar to those in the previous section show that, assuming always that $\theta>1/4$, the scheme is stable with thresholds $\mu h^{1/2}$ for arbitrary refinements $k=\sigma(h)$. On the other hand the general Theorem 4.1, cannot be applied to the case at hand assuming only $k=\sigma(h)$, because we need that the local errors behave as $o(R_h)$. Since (5.1) is clearly second order accurate in space and time, the hypotheses of the Theorem 4.1 are satisfied for grid refinements $k=sh^\delta$, s and δ constant, $s>0$, $\delta>1/4$. More precisely, the application of the Theorem 4.1 to (5.1) reads as follows:

Theorem 5.1. *Assume that $\theta > 1/4$ and that:*

(i) *For $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$, the solution of* (2.1)–(2.4) *possesses (bounded) continuous derivatives up to the sixth order.*

(ii) *The grid is refined according $k = sh^{\delta}$, s and δ constant, $s > 0$, $\delta > 1/4$.*

(iii) *The discrete initial data satisfy* (4.5).

Then for h sufficiently small the (5.1) possess a solution $\mathbf{U}^n$, $n = 0, 1, \ldots, N$, with

$$\max\{\|(\mathbf{U}^{n+1} - \mathbf{r}_h u^{n+1}, \mathbf{U}^n - \mathbf{r}_h u^n)\|_E^*, n = 0, 1, \ldots, N-1\} = \mathcal{O}(k^2 + h^2), \qquad h \to 0.$$

As in Theorem 4.4, the energy estimates lead to estimates of the same order in the Sobolev norm (5.4).

When $\theta < 1/4$, a standard von Neumann analysis of the principal part shows that (5.1) is not unconditionally stable. For the limit case $\theta = 1/4$, not covered so far, (5.3) must be replaced by

$$\begin{aligned} Q_k^*(\mathbf{W}, \mathbf{W}^*) &\leqq \|(\mathbf{W} - \mathbf{W}^*)/k\|^2 + (1/2)\|D^2\mathbf{W}\|^2 + (1/2)\|D^2\mathbf{W}^*\|^2 \\ &\leqq (1 + r^2) Q_k^*(\mathbf{W}, \mathbf{W}^*), \qquad (\mathbf{W}, \mathbf{W}^*) \in \mathbb{Z}_h \times \mathbb{Z}_h. \end{aligned}$$

These inequalities, which are established by using the technique in the proof of the Proposition 3.1, reveal that in this case the energy norm is equivalent to (5.4) uniformly in h, provided that k is restricted as $k/h^2 =$ arbitrary constant. The previous theorem can be shown to hold also for $\theta = 1/4$ under the additional restriction that $\delta \geqq 2$. Within the range of unconditional stability $\theta \geqq 1/4$, the value 1/4 is of particular interest because leads to the smallest time truncation error.

6 Numerical experiments

The schemes analised above have been tested in the long-time integration of solitary waves and collision of solitary waves. Equation (1.1) shows that this kind of solution decays exponentially as $|x| \to \infty$ and therefore, for numerical purposes we have employed the schemes on an interval (x_L, x_R), where the artificial boundaries x_L and x_R are located far enough for the theoretical solution to satisfy the periodic boundary conditions, except for a negligible remainder. For both schemes the starting data were chosen according to (4.4). When using the implicit schemes, the nonlinear system of equations to be solved at each time level takes the form

$$\mathbb{A}\mathbf{U}^{n+1} = \mathbb{B}\mathbf{U}^n + \mathbb{C}\mathbf{U}^{n-1} + \mathbb{D}[\theta(\mathbf{U}^{n+1})^2 + (1-2\theta)(\mathbf{U}^n)^2 + \theta(\mathbf{U}^{n-1})^2],$$

where $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ are circulant matrices with five nonzero elements per row. Furthermore these entries depend on θ, k and h, but not on n. This suggests the fixed-point iteration

$$\mathbb{A}\mathbf{U}_{s+1}^{n+1} = \mathbb{B}\mathbf{U}^n + \mathbb{C}\mathbf{U}^{n-1} + \mathbb{D}[\theta(\mathbf{U}_s^{n+1})^2 + (1-2\theta)(\mathbf{U}^n)^2 + \theta(\mathbf{U}^{n-1})^2],$$

where the only matrix to be inverted in each time-integration is $\mathbb{A}$. The initial guess $\mathbf{U}_0^{n+1}$ is computed from $\mathbf{U}^n$, $\mathbf{U}^{n-1}$ by means of the explicit scheme.

Table 1

h	k	Error	CPU
1.0	0.4	0.0070	4
	0.2	0.0071	11
	0.1	0.0072	18
0.5	0.1	0.0017	33
	0.05	0.0017	63
	0.025	0.0018	123
0.25	0.025	0.0004	237
	0.0125	0.0004	472
	0.00625	0.0004	941

Table 2

h	k	Error	CPU	Iter
1.0	0.4	0.0080	32	3
	0.2	0.0074	45	2
	0.1	0.0073	75	1
0.5	0.4	0.0031	64	3
	0.2	0.0021	101	2
	0.1	0.0018	124	1
0.025	0.4	0.0020	134	3
	0.2	0.0009	204	2
	0.1	0.0006	240	1

Tables 1 and 2 refer to a single-soliton solution and correspond, respectively, to the explicit method and to the implicit method with $\theta=1/4$. The value $\theta=1/3$ was also tested, but the results are not reported here, as they are very similar to those obtained with $\theta=1/4$. The theoretical solution has an amplitude $A=0.5$ and an initial phase $\xi_0=0$. The boundaries are located at $x_L=-60$, $x_R=60$ and the integration was followed up to $T=40$. The tables display information at $t=2$. The column "error" shows $\|\mathbf{U}^n-\mathbf{r}_h u^n\|_\infty$ and CPU refers to the CPU time in hundredths of a second on a VAX-11/780 machine with a VAX-11 FORTRAN compiler. For the implicit scheme the last column gives the average number of inner iterations per time-step. The computations were carried out in single precision and the inner iteration of the implicit scheme was stopped when two consecutives iterates were found which in the discrete L^2-norm differed in less than 10^{-5}.

The expected rates of convergence show up in the tables. Note that in the explicit scheme a reduction in k with h fixed does not change the error. This proves that for the value of k allowed by the stability restriction $k<0.5h^2$ the integration in time is very accurate and the error originates, almost entirely, from the space discretization (i.e. the results given by the scheme are the same as those of the corresponding time-continuous, space-discretized method).

A comparison of both tables reveals that the explicit method is more efficient than its implicit counterparts. This is due to the fact that the longer time steps that can be used in the implicit algorithms do not make up for the higher cost of performing an implicit time step. It is fair to say that this conclusion is specific to the periodic boundary conditions used here. We have also run the schemes with homogeneous Dirichlet conditions $u=u_x=0$, for which the linear algebra is somewhat simpler than for the periodic case (the matrix to be inverted is pentadiagonal). With such alternative boundary conditions and small values of h, the implicit schemes are more efficient than the explicit method.

We recall that, even though the tables correspond to $t=2$, the integration was followed up to $T=40$. The errors exhibit a roughly linear growth with t and no problems of nonlinear blow-up were encountered.

Numerical results corresponding to soliton interactions can be seen in [8].

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